AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **24**, 742 (1983); doi: 10.1063/1.525790
View online: http://dx.doi.org/10.1063/1.525790
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/24/4?ver=pdfcov
Published by the AIP Publishing

Some character theory for groups of linear and antilinear operators

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(Received 5 October 1981; accepted for publication 20 November 1981)

Elementary group concepts are recast into a form applicable to finite magnetic groups of linear and antilinear operators. Analogs of useful definitions for linear groups such as the Frobenius–Schur invariant, commutator subgroups, and ambivalent classes are considered. These are applied to the 180 magnetic single and double point groups and it is shown that only seven require independent treatment of characters.

PACS numbers: 02.20. + b

1. INTRODUCTION

The use of group theory in certain areas of physics and chemistry is now well established. This generally proceeds through some form of representation theory (vector representations, ray representations, vector corepresentations, or ray corepresentations) of a group of operators on a Hilbert space.[1,2] The form used depends critically on the nature of the operators, as to whether they are linear or antilinear (Wigner[3] has shown that only these two types of operator need be considered in quantum mechanics). For vector representations of groups of linear operators an extensive literature exists, with contributions from mathematicians, physicists and chemists. Qualitative applications of vector representations (such as selection rules) are based on character theory[4,5] whereas semiquantitative calculations through the Wigner–Eckart theorem use both basis dependent information in the *n-jm* symbols[6,7] and character theory in the *n-j* symbols and isoscalars.[8,9] Characters, of course, need not be considered as any information obtainable from them can also be obtained from any realization of the vector representation, but their use enormously simplifies many calculations and justifies their detailed considerations.

Surprisingly, the character theory for the other types of representation is extremely ill-developed. Backhouse[10] has shown that a character table exists for ray representations of finite groups and Newmarch and Golding[11] (henceforth denoted as N-G) for the vector corepresentations of finite magnetic groups of linear and antilinear operators, while even this is missing for ray corepresentations. Standard vector representation concepts such as the Frobenius–Schur invariant do not appear to have been considered. In part the purpose of this paper is to fill in some of these gaps for the vector corepresentations of finite magnetic groups by considering one-dimensional irreducible corepresentations (ICRs), faithful ICRs, and complex conjugates of ICRs (Secs. 6 and 8).

During the course of this investigation an even more important gap became apparent. Magnetic groups are rather special groups in that they possess a certain subgroup of index two. This subgroup corresponds to the linear operators and its coset to the antilinear operators. The subgroup is obviously fixed by physical considerations and linear operators cannot be changed into antilinear ones without changing their physical applicability. This is reflected in the mathematics of corepresentations, and shows that a magnetic group must be considered as a *pair* of groups. The following three sections are devoted to the elementary group theory of this situation where isomorphism, homomorphism, etc., are discussed. Our own opinion is that much of the material of these sections should be self-evident. However, inappropriate statements—particularly in regard to isomorphism—have appeared sufficiently often to prompt us to spell them out. This vein is followed in Sec. 5, where it is shown that direct products of magnetic groups can be usefully defined.

The third aim of this paper is to reduce the number of magnetic single and double point groups (180 in all) requiring separate treatment of characters. Assuming known character theory of linear groups, by isomorphism (Sec. 2), factor groups (Sec. 3), direct products (Sec. 5), and an examination of the intertwining numbers (Sec. 7), it is shown that only seven groups need be considered.

Examples are drawn from the finite magnetic point groups. The theory is applicable by finite approximations to the magnetic space groups,[1] the spin groups,[12,13] and the line groups of stereo-regular polymers.[14] Much is readily transferable to compact groups, where it should find applications due to the PCT theorem for elementary particles.[15]

In general the notation is that of N-G. Magnetic point groups are labelled as in Bradley and Cracknell,[1] with an asterisk to denote the double groups. The ICRs of these groups are also labelled as in Bradley and Cracknell,[1] save for the typographical omission of the prefix D when there is no increase in degeneracy in inducing ICRs from the linear subgroup. E or e denotes the identity of the group, or the unit matrix. A prefix M denotes the magnetic group analog of a linear group concept. Proofs are usually omitted whenever they are simple modifications of those for linear groups.

2. ISOMORPHISMS AND HOMOMORPHISMS

It is only the physical importance of the time reversal operator which leads to the study of magnetic groups. Such a group contains a subgroup of linear operators and a coset of antilinear operators and clearly, to maintain their applicability, we cannot arbitrarily change linear operators into antilinear ones or vice versa. The subgroup of linear operators is just as important as the group itself. An abstract definition which indicates this is

Definition 2.1: A magnetic group M is an ordered pair of groups $M = (G,H)$ where H has index two in G.

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 0022-2488/83/040742-15$02.50

Classifications of groups into families are made according to various criteria. For example, there is the equivalence family of D_2, consisting of all point groups mapped onto one another by automorphisms of $O(3)$. Such an equivalence concept gives 90 families of grey and nongrey magnetic single point groups and a further 90 families of double groups.[1] In addition the fundamental group concept of isomorphism may be applied to linear groups to reduce, say, the 32 families of crystallographically distinct point groups down to 11 nonisomorphic families. However, all statements sighted on "isomorphic magnetic groups" have been rather misleading [$M_1 = (G_1, H_1)$ is isomorphic to $M_2 = (G_2, H_2)$ if $G_1 \cong G_2$] as the position of the linear subgroup need not be preserved. For example, $6'22'$ and $62'2'$ have ICRs of different dimension so degeneracies cannot be transferred despite $G_1 \cong G_2$. An appropriate definition is

Definition 2.2: Two magnetic groups $M_1 = (G_1,H_1)$ and $M_2 = (G_2,H_2)$ are M-isomorphic iff there is a group isomorphism $\phi: G_1 \to G_2$ for which $\phi(H_1) = H_2$.

This is a very stringent condition and generally requires explicit construction of the isomorphism. It cannot, for example, be weakened to *an* isomorphism $\phi: G_1 \to G_2$ and another from H_1 to H_2. To see this, consider the group $16\Gamma_2 c$ of order 16 with presentation $\langle x,y | x^4 = y^4 = e, xy = yx^3 \rangle$ from the tables of Hall and Senior.[16] This group contains $(21) = Z_4 \otimes Z_2$ once characteristically (i.e., invariant under all automorphisms of $16\Gamma_2 c_2$) and twice noncharacteristically. Setting $G_1 = G_2 = 16\Gamma_2 c_2$, H_1 the characteristic subgroup and H_2 one of the noncharacteristic ones, then there is no M-isomorphism of M_1 onto M_2 (which here would be an automorphism) despite G_1 and G_2,H_1 and H_2 being pairwise isomorphic. The two magnetic groups are essentially different. (In fact, the first has seven ICRs and the second has eight.) A calculation for the 180 single and double magnetic point groups yields 64 nonisomorphic families which are collected in Table I.

We have dwelt on the concept of isomorphism at length primarily to show that a magnetic group must be considered as a pair of groups. These should now be obvious:

Definition 2.3: An M-homomorphism ϕ of $M_1 = (G_1,H_1)$ into $M_2 = (G_2,H_2)$ is a homomorphism ϕ of G_1 into G_2 such that $\phi(H_1) \subseteq H_2$ and $\phi(G_1 - H_1) \subseteq G_2 - H_2$.

This ensures that linear elements are mapped onto linear elements and antilinear onto antilinear. This definition has been used by Janssen in discussing projective corepresentations.[17]

Definition 2.4: An M-normal subgroup of $M = (G,H)$ is a subgroup of H (and hence of G) which is normal in G (and hence normal in H).

The subgroups of G for the magnetic single point groups have been listed by Ascher and Janner,[18] and of course only a few are M-normal. Later it is shown that they may be obtained from the character table. For the moment,

Theorem 2.5 (First Isomorphism Theorem): Let $M = (G,H)$ be a magnetic group and ϕ an M-homomorphism of M. Then the kernel of ϕ is an M-normal subgroup L and the image of M is naturally M-isomorphic to $(G/L, H/L)$. Conversely, each M-normal subgroup L defines an M-homomorphism of M onto $(G/L, H/L)$.

The other isomorphism theorems can be similarly adapted. However, this is all we need for now.

3. COREPRESENTATIONS

Definition 3.1: A corepresentation D is an M-homomorphism of a magnetic group into a magnetic group of operators (G,H) over a complex vector space, where the operators of H are linear and of G-H are antilinear.

Herbut *et al.*[19] have given a similar definition for their unitary/antiunitary representations of magnetic groups and introduced the term "antimatrix" for the matrix of an antilinear operator. Whilst we support their viewpoint in which sense we have interpreted corepresentations, we consider that the tensor notation from spinor calculus used by Newmarch and Golding[20] handles antilinear operators in the most effective manner. We regard both the common notation used here and that of Herbut *et al.*[19] to be "approximations" to the tensor notation, and use the common notation on the grounds of familiarity and a mild preference for seeing complex conjugates explicitly.

Matrices of linear and antilinear operators of a corepresentation and irreducible corepresentations (ICRs) are defined in the normal way. From these we have

Lemma 3.2: Let D be a corepresentation of $M = (G,H)$ with character χ and let $u \epsilon H$. If n is the order of u and f the degree of D [$f = \chi(e)$] then

(a) $D(u)$ is similar to diag. $(\epsilon_1, \epsilon_2, \ldots, \epsilon_f)$,

(b) $\epsilon_i^n = 1$ for all i,

(c) $\chi(u) = \sum_{I=1}^{f} \epsilon_i$,

(d) $|\chi(u)| \leqslant \chi(e) = f$.

Lemma 3.3: If D is a corepresentation of M, then the kernel of D ($\ker D$) is an M-normal subgroup of M, and $u \epsilon \ker D$ iff $\chi(u) = \chi(e)$.

Lemma 3.4: Let $D = \Sigma n_i D_i$ be a corepresentation of M and D_i be ICRs. Then $\ker D = \cap \{\ker D_i : n_i > 0\}$ and $\cap \{\ker D_i : \text{all ICRs}\} = \{e\}$.

These are all proved in exactly the same manner as for representations (e.g., Isaacs[21]). The regular corepresentation and its properties are given in N-G.

Every M-normal subgroup of a magnetic group may be found from the character table by taking irreducible characters and sums of characters and finding those elements u for which $\chi(u) = \chi(e)$. For example, the group $4'/mmm$ has M-normal subgroups $\{E,I\}$ from E_g, $\{E,C_{2z},\sigma_z\}$ from B_{1g}, $\{E,C_{2x},C_{2y},C_{2z}\}$ from A_u, $\{E,\sigma_z\}$ from E_u, $\{E,C_{2z},\sigma_x,\sigma_y\}$ from B_{1u}, and $\{E,C_{2z}\}$ from $B_{1g} \oplus A_u$. (The character table is given in N-G).

One of the major features which distinguishes corepresentation theory from representation theory is the different form of Schur's lemmas for the two theories. For linear groups any matrix commuting with an IR is a constant diagonal matrix (quantitatively, the set of all such matrices form an algebra of dimension one over $\mathbb{C}$). In N-G it was shown

TABLE I. The M-isomorphic families of magnetic point groups. The families are listed by ascending orders of the groups. The notation for groups and group elements is that of Bradley and Cracknell[1] with an asterisk to distinguish double groups. Elements isomorphic to each other in each family are listed in the same order in rows of the "Isomorphism" column. A point group label is given for G under "Popular name for G" although for double groups, $\theta^2 = \bar{E}$ and G is not in fact the point group. A "dash" here indicates a grey group. The comments are illustrative, not exhaustive.

Family	Order	Magnetic group	Popular name for G	H	Isomorphism	Comments
1	2	$11'$	C_1'	C_1	θ	Character
		$\bar{1}'$	C_i	C_1	θI	table as
		$2'$	C_2	C_1	θC_{2z}	H
		m'	C_{1h}	C_1	$\theta\sigma_h$	
2	4	$22'2'$	D_2	C_2	$C_{2z},\theta C_{2y}$	Character
		$2/m'$	C_{2h}	C_2	$C_{2z},\theta I$	table as
		$21'$	C_2'	C_2	C_{2z},θ	H
		$2'/m'$	C_{2h}	C_i	$I,\theta C_{2z}$	
		$2'/m$	C_{2h}	C_{1h}	$\sigma_z,\theta I$	
		$\bar{1}1'$	C_i'	C_i	I,θ	
		$^*m'$	C_{1h}^*	C_1^*	$\bar{E},\theta\sigma_z$	
		$^*2'$	C_2^*	C_1^*	$\bar{E},\theta C_{2z}$	
		$2m'm'$	C_{2v}	C_2	$C_{2z},\theta\sigma_y$	
		$2'm'm$	C_{2v}	C_{1h}	$\sigma_y,\theta C_{2x}$	
		$m1'$	C_{1h}'	C_{1h}	σ_z,θ	
3	4	$4'$	C_4	C_2	θC_{4z}^+	Homomorphic
		$\bar{4}'$	S_4	C_2	θS_{4z}^-	image of
		$^*11'$	$C_1^{*\prime}$	C_1^*	θ	family 8
		$^*\bar{1}'$	C_i^*	C_1^*	θI	
4	6	$32'$	D_3	C_3	$C_3^+,\theta C_{21}'$	Character table
		$\bar{3}m'$	D_{3d}	S_6	$C_3^+,\theta\sigma_{d1}$	as H
5	6	$6'$	C_6	C_3	θC_6^+	Homomorphic
		$\bar{6}'$	C_{3h}	C_3	θS_3^-	image of,
		$\bar{3}'$	S_6	C_3	θS_6^+	e.g., family 13
		$31'$	C_3'	C_3	θC_3^+	
6	8	$4/m'$	C_{4h}	C_4	$C_{4z}^+,\theta I$	Homomorphic
		$4'/m'$	C_{4h}	S_4	$S_{4z}^-,\theta I$	image of,
		$^*m1'$	$C_{1h}^{*\prime}$	C_{1h}^*	σ_z,θ	e.g., family 23
		$^*2'/m$	C_{2h}^*	C_{1h}^*	$\sigma_z,\theta I$	
		$^*21'$	$C_2^{*\prime}$	C_2^*	C_{2z},θ	
		$41'$	C_4'	C_4	C_{4z}^+,θ	
		$\bar{4}1'$	S_4'	S_4	S_{4z}^-,θ	
		$^*2/m'$	C_{2h}^*	C_2^*	$C_{2z},\theta I$	
7	8	$4'22'$	D_4	D_2	$C_{2x},\theta C_{4z}^+$	Homomorphic
		$\bar{4}'2m'$	D_{2d}	D_2	$C_{2x},\theta S_{4z}^-$	image of
		$4'mm'$	C_{4v}	C_{2v}	$\sigma_x,\theta C_{4z}^+$	family 19
		$\bar{4}'2'm$	D_{2d}	C_{2v}	$\sigma_x,\theta S_{4z}^-$	
		$^*\bar{1}1'$	$C_i^{*\prime}$	C_i^*	I,θ	
8	8	$^*4'$	C_4^*	C_2^*	θC_{4z}^+	
		$^*\bar{4}'$	S_4^*	C_2^*	θS_{4z}^{*-}	
9	8	$42'2'$	D_4	C_4	$C_{4z}^+,\theta C_{2x}$	Character
		$4m'm'$	C_{4v}	C_4	$C_{4z}^+,\theta\sigma_x$	table as H
		$\bar{4}2'm'$	D_{2d}	S_4	$S_{4z}^-,\theta C_{2x}$	
		$^*2'2'2$	D_2^*	C_2^*	$C_{2z},\theta C_{2y}$	
		$^*m'm'2$	C_{2v}^*	C_2^*	$C_{2z},\theta\sigma_y$	
		$^*m'm2'$	C_{2v}^*	C_{1h}^*	$\sigma_y,\theta\sigma_x$	
10	8	$2221'$	D_2'	D_2	C_{2x},C_{2y},θ	Character
		$m'm'm'$	D_{2h}	D_2	$C_{2x},C_{2y},\theta I$	table as H
		$mm21'$	C_{2v}'	C_{2v}	σ_x,σ_y,θ	
		mmm'	D_{2h}	C_{2v}	$\sigma_x,\sigma_y,\theta I$	
		$m'm'm$	D_{2h}	C_{2h}	$C_{2z},I,\theta C_{2y}$	

TABLE I. (*Continued*)

Family	Order	Magnetic group	Popular name for G	H	Isomorphism	Comments
		$^*2'/m'$	C_{2h}^{*}	C_i^{*}	$I,\bar{E},\theta C_{2z}$	
		$2/m1'$	C'_{2h}	C_{2h}	C_{2z},I,θ	
11	8	$4'/m$	C_{4h}	C_{2h}		Direct product of $\bar{1}1'$ with $4'$ (family 3)
12	12	$^*31'$	$C_3^{*\prime}$	C_3^{*}	C_3^+,θ	
		$^*\bar{3}'$	S_6^{*}	C_3^{*}	$C_3^+,\theta I$	
13	12	$^*6'$	C_6^{*}	C_3^{*}	$C_3^+,\theta C_2$	Direct product of $\bar{1}1'$ with $6'$ (family 5)
		$^*\bar{6}'$	C_{3h}^{*}	C_3^{*}	$C_3^+,\theta\sigma_h$	
		$6/m'$	C_{6h}	C_6	$C_6^+\theta\sigma_h$	
		$6'/m$	C_{6h}	C_{3h}	$S_3^-,\theta\sigma_h$	
		$6'/m'$	C_{6h}	S_6	$S_6^-,\theta\sigma_h$	
14	12	$^*3m'$	C_{3v}^{*}	C_3^{*}	$C_3^+,\theta\sigma_{d1}$	Character table as H
		$^*32'$	D_3	C_3^{*}	$C_3^+,\theta C'_{21}$	
15	12	$\bar{3}m'$	D_{3d}	S_6		Character table as H
16	12	$6'22'$	D_6	D_3	$C_3^+,C'_{21},\theta C_2$	Character table as H
		$6'mm'$	C_{6v}	C_{3v}	$C_3^+,\sigma_{d1},\theta C_2$	
		$\bar{6}'m'2$	D_{3h}	D_3	$C_3^+,C'_{21},\theta\sigma_h$	
		$\bar{6}'m2'$	D_{3h}	C_{3v}	$C_3^+,\sigma_{v1},\theta\sigma_h$	
		$\bar{3}'m$	D_{3d}	D_3	$C_3^+,C'_{21},\theta I$	
		$\bar{3}'m$	D_{3d}	C_{3v}	$C_3^+,\sigma_{d1},\theta I$	
		$321'$	D'_3	D_3	C_3^+,C'_{21},θ	
		$3m1'$	C'_{3v}	C_{3v}	C_3^+,σ_{d1},θ	
17	12	$62'2'$	D_6	C_6	$C_6^+,\theta C'_{21}$	Character table as H
		$6m'm'$	C_{6v}	C_6	$C_6^+,\theta\sigma_{d1}$	
		$\bar{6}m'2'$	D_{3h}	C_{3h}	$S_3^-,\theta C'_{21}$	
18	12	$\bar{3}1'$	S'_6	S_6	S_6^-,θ	Homomorphic image of, e.g., family 29
		$61'$	C'_6	C_6	C_6^+,θ	
		$\bar{6}1'$	C'_{3h}	C_{3h}	S_3^-,θ	
19	16	$^*4'22'$	D_4^{*}	D_2^{*}	$C_{2x},C_{2y},\theta C_{4z}^+$	
		$^*\bar{4}'2m'$	D_{2d}^{*}	D_2^{*}	$C_{2x},C_{2y},\theta S_{4z}^-$	
		$^*4'mm'$	C_{4v}^{*}	C_{2v}^{*}	$\sigma_x,\sigma_y,\theta C_{4z}^+$	
		$^*\bar{4}'m2'$	D_{2d}^{*}	C_{2v}^{*}	$\sigma_x,\sigma_y,\theta S_{4z}^-$	
20	16	$^*4m'm'$	C_{4v}^{*}	C_4^{*}	$C_{4z}^-,\theta\sigma_x$	Character table as H
		$^*42'2'$	D_4^{*}	C_4^{*}	$C_{4z}^-,\theta C_{2x}$	
		$^*\bar{4}2'm'$	D_{2d}^{*}	S_4^{*}	$S_{4z}^+,\theta C_{2x}$	
21	16	$^*2221'$	$D_2^{*\prime}$	D_2^{*}	C_{2x},C_{2y},θ	Character table as H
		$^*mm21'$	$C_{2v}^{*\prime}$	C_{2v}^{*}	σ_x,σ_y,θ	
		$^*m'm'm'$	D_{2h}^{*}	D_2^{*}	$C_{2x},C_{2y},\theta I$	
		$^*mmm'$	D_{2h}^{*}	C_{2v}^{*}	$\sigma_x,\sigma_y,\theta I$	
22	16	$4/m'm'm'$	D_{4h}	D_4	$C_{4z}^+,C_{2x},\theta I$	Character table as H
		$4/m'mm$	D_{4h}	C_{4v}	$C_{4z}^+,\sigma_x,\theta I$	
		$4'/m'm'm$	D_{4h}	D_{2d}	$S_{4z}^-,C_{2x},\theta I$	
		$4221'$	D'_4	D_4	C_{4z}^+,C_{2x},θ	
		$\bar{4}2m1'$	D'_{2d}	D_{2d}	S_{4z}^-,C_{2x},θ	
		$4mm1'$	C'_{4v}	C_{4v}	C_{4z}^+,σ_x,θ	
23	16	$^*4/m'$	C_{4h}^{*}	C_4^{*}	$C_{4z}^+,\theta I$	
		$^*41'$	$C_4^{*\prime}$	C_4^{*}	C_{4z}^+,θ	
		$^*4'/m'$	C_{4h}^{*}	S_4^{*}	$S_{4z}^-,\theta I$	

TABLE I. (*Continued*)

Family	Order	Magnetic group	Popular name for G	H	Isomorphism	Comments
		$*\bar{4}1'$	$S_4^{*\prime}$	S_4^*	S_{4z}^-,θ	
24	16	$*4'/m$	C_{4h}^*	C_{2h}^*		Direct product of $\bar{1}1'$ with $*4'$ (family 8)
25	16	$*m'm'm$	D_{2h}^*	C_{2h}^*	$C_{2z},I,\theta C_{2x}$	Character table as H
		$4/mm'm'$	D_{4h}	C_{4h}	$C_{4z}^+,I,\theta C_{2x}$	
26	16	$*2/m1'$	$C_{2h}^{*\prime}$	C_{2h}^*	C_{2z},I,θ	Direct product of $\bar{1}1'$ with $41'$ (family 6)
		$4/m1'$	C_{4h}'	C_{4h}	$C_{4z}^+,I,\theta C_{4z}^+$	
27	16	$4'/mmm$	D_{4h}	D_{2h}		Homomorphic image of family 42
28	16	$mmm1'$	D_{2h}'	D_{2h}		Character table as H
29	24	$*61'$	$C_6^{*\prime}$	C_6^*	C_6^+,θ	
		$*6/m'$	C_{6h}^*	C_6^*	$C_6^+,\theta I$	
		$*6'/m$	C_{6h}^*	C_{3h}	$S_3^-,\theta I$	
		$*\bar{6}1'$	$C_{3h}^{*\prime}$	C_{3h}^*	S_3^-,θ	
30	24	$*3m1'$	$C_{3v}^{*\prime}$	C_{3v}^*	C_3^+,θ_{d1},θ	
		$*\bar{3}'m$	D_{3d}^*	C_{3v}^*	$C_3^+,\sigma_{d1},\theta I$	
		$*321'$	$D_3^{*\prime}$	D_3^*	C_3^+,C_{21}',θ	
		$*\bar{3}'m'$	D_{3d}^*	D_3^*	$C_3^+,C_{21}',\theta I$	
31	24	$*62'2'$	D_6^*	C_6^*	$C_6^+,\theta C_{21}'$	Character table as H
		$*6m'm'$	C_{6v}^*	C_6^*	$C_6^+,\theta\sigma_{d1}$	
		$*\bar{6}m'2'$	D_{3h}^*	C_{3h}^*	$S_3,\theta C_{21}'$	
32	24	$*\bar{3}m'$	D_{3d}^*	S_6^*		Character table as H
33	24	$*6'2'2$	D_6^*	D_3^*	$C_3^+,C_{21}',\theta C_2$	Character table as H
		$*6'm'm$	C_{6v}^*	C_{3v}^*	$C_3^+,\sigma_{d1},\theta C_2$	
		$*\bar{6}'m'2$	D_{3h}^*	D_3^*	$C_3^+,C_{21}',\theta\sigma_h$	
		$*\bar{6}'m2'$	D_{3h}^*	C_{3v}^*	$C_3^+,\sigma_{v1},\theta\sigma_h$	
34	24	$*6'/m'$	C_{6h}^*	S_6^*		Direct product of $\bar{1}1'$ with $*6'$ (family 13)
35	24	$*\bar{3}1'$	$S_6^{*\prime}$	S_6^*		Direct product of $\bar{1}1'$ with $*31'$ (family 12)
36	24	$6/m1'$	C_{6h}'	C_{6h}		Direct product of $\bar{1}1'$ with $61'$ (family 12)
37	24	$6'/m'm'm$	D_{6h}	D_{3d}	$S_6^-,C_{21}',\theta C_2$	Character table as H
		$6'/mm'm$	D_{6h}	D_{3h}	$S_3^-,C_{21}',\theta I$	
		$6/m'mm$	D_{6h}	C_{6v}	$C_6^+,\sigma_{d1},\theta I$	
		$6221'$	D_6'	D_6	C_6^+,C_{21}',θ	
		$6mm1'$	C_{6v}'	C_{6v}	C_6^+,σ_{d1},θ	
		$\bar{6}2m1'$	D_{3h}'	D_{3h}	C_3^-,C_{21}',θ	
		$\bar{3}m1'$	D_{3d}'	D_{3d}	S_6^-,C_{21}',θ	
38	24	$6/mm'm'$	D_{6h}	C_{6h}		Character table as H

TABLE I. (*Continued*)

39	24	231′	T'	T	C_{31}^-,C_{2x},θ	Homomorphic image
		$m'3$	T_h	T	$C_{31}^-,C_{2x},\theta I$	of family 53
40	24	$\bar{4}'3m'$	T_d	T	$C_{31}^-,C_{2x},\theta\sigma_{da}$	Character table
		4′32′	0	T	$C_{31}^-,C_{2x},\theta C_{2a}$	as H
41	32	$4/mmm1'$	D'_{4h}	D_{4h}		Character table as H
42	32	$*4'/mmm'$	D^*_{4h}	D^*_{2h}		Direct product of $\bar{1}1'$ with *4′22′ (family 19)
43	32	$*4/mm'm'$	D^*_{4h}	C^*_{4h}		Character table as H
44	32	$*4/m'm'm'$	D^*_{4h}	D^*_4	$C^+_{4z},C_{2x},\theta I$	Character table
		$*4/m'mm$	D^*_{4h}	C^*_{4v}	$C^+_{4z},\sigma_x,\theta I$	as H
		$*4'/m'm'm$	D^*_{4h}	D^*_{2d}	$S^-_{4z},C_{2x},\theta I$	
		*4221′	$D^{*\prime}_4$	D^*_4	C^+_{4z},C_{2x},θ	
		$*4mm1'$	$C^{*\prime}_{4v}$	C^*_{4v}	C^+_{4z},σ_x,θ	
		$*\bar{4}2m1'$	$D^{*\prime}_{2d}$	D^*_{2d}	S^-_{4z},C_{2x},θ	
45	32	$*mmm1'$	$D^{*\prime}_{2h}$	D^*_{2h}		Character table as H
46	32	$*4/m1'$	$C^{*\prime}_{4h}$	C^*_{4h}		Direct product of $\bar{1}1'$ with *41′ (family 23)
47	48	$*6'/m'm'm$	D^*_{6h}	D^*_{3d}		Character table as H
48	48	$*6/mm'm'$	D^*_{6h}	C^*_{6h}		Character table as H
49	48	$*6'/mmm'$	D^*_{6h}	D^*_{3h}	$S^-_3,C'_{21},\theta I$	Character table
		$*6/m'm'm'$	D^*_{6h}	D^*_6	$C^+_6,C'_{21},\theta I$	as H
		$*6/m'mm$	D^*_{6h}	C^*_{6v}	$C^+_6,\sigma_{d1},\theta I$	
		*6221′	$D^{*\prime}_6$	D^*_6	C^+_6,C'_{21},θ	
		$*6mm1'$	$C^{*\prime}_{6v}$	C^*_{6v}	C^+_6,σ_{d1},θ	
		$*\bar{6}2m1'$	$D^{*\prime}_{3h}$	D^*_{3h}	S^-_3,C'_{21},θ	
50	48	$6/mmm1'$	D'_{6h}	D_{6h}		Character table as H
51	48	$*m'3$	T^*_h	T^*	$C^-_{31},C_{2x},\bar{C}_{2y},\theta I$	
		*231′	$T^{*\prime}$	T^*	$C^-_{31},C_{2x},\bar{C}_{2y},\theta$	
52	48	$*\bar{4}'3m'$	T^*_d	T^*	$C^-_{31},C_{2x},\bar{C}_{2y},\theta\sigma_{da}$	Character
		*4′32′	0*	T^*	$C^-_{31},C_{2x},\bar{C}_{2y},\theta C_{2a}$	table as H
53	48	$*\bar{3}m1'$	$D^{*\prime}_{3d}$	D^*_{3d}		Direct product of $\bar{1}1'$ with *321′ (family 30)
54	48	$*6/m1'$	$C^{*\prime}_{6h}$	C^*_{6h}		Direct product of $\bar{1}1'$ with *61′ (family 29)
55	48	$m'3m'$	0_h	0	$C^+_{4x},C^-_{31},C_{2b},\theta I$	Character table
		4321′	$0'$	0	$C^+_{4x},C^-_{31},C_{2b},\theta$	as H
		$m'3m$	0_h	T_d	$S^-_{4x},C^-_{31},\sigma_{db},\theta I$	
		$\bar{4}3m1'$	T'_d	T_d	$S^-_{4x},C^-_{31},\sigma_{db},\theta$	
56	48	$m31'$	T'_h	T_h		Homomorphic image of family 62

TABLE I. (*Continued*)

57	48	$m3m'$	0_h	T_h		Character table as H
58	64	$*4/mmm1'$	$D_{4h}^{*\prime}$	D_{4h}^{*}		Character table as H
59	96	$*m'3m'$	0_h^*	0^*	$C_{4x}^+,\bar{C}_{31}^-,C_{2b},\theta I$	Character table as H
		$*m'3m$	0_h^*	T'_d	$S_{4x}^-,\bar{C}_{31}^-,\sigma_{db},\theta I$	
		$*4321'$	$0^{*\prime}$	0^*	$C_{4x}^+,\bar{C}_{31}^-,C_{2b},\theta$	
		$*\bar{4}3m1'$	$T_d^{*\prime}$	T_d^*	$S_{4x}^-,\bar{C}_{31}^-,\sigma_{db},\theta$	
60	96	$*m31'$	$T_h^{*\prime}$	T_h^*		Direct product of $\bar{1}1'$ with $231'$ (family 53)
61	96	$*m3m'$	0_h^*	T_h^*		Character table as H
62	96	$m3m1'$	$0'_h$	0_h		Character table as H
63	96	$*6/mmm1'$	$D_{6h}^{*\prime}$	D_{6h}^{*}		Direct product of $\bar{1}1'$ with $*6221'$ (family 51)
64	192	$*m3m1'$	$0_h^{*\prime}$	0_h^*		Character table as H

that the algebra of such matrices is of dimension one, four or two over $\mathbb{R}$ (i.e., is isomorphic to $\mathbb{R}$,$\mathbb{Q}$, or $\mathbb{C}$). Labelling the ICRs as types (a), (b), and (c), respectively, in concordance with standard usage, the intertwining number I was introduced: for an ICR of type (a), $I = 1$, for an ICR of type (b), $I = 4$, and for an ICR of type (c), $I = 2$. The row orthogonality relation for ICRs was then shown to be

$$\sum_u \chi_i(u)\chi_j(u)^* = \delta_{ij} I_i |H|.$$

These next results all follow as for representation theory (e.g., Isaacs[21]).

Theorem 3.5: Let D be a corepresentation of $M = (G,H)$ and $L \subseteq \ker D$ an M-normal subgroup of M. Define $\hat{D}$ on M/L by $\hat{D}(gL) = D(g)$ for all $g \in G$. Then

(a) $\hat{D}$ is a corepresentation of M/L,

(b) $\hat{D}$ is irreducible iff D is irreducible,

(c) if D is irreducible with intertwining number I and $\hat{D}$ has intertwining number $\hat{I}$, $I = \hat{I}$.

Conversely,

Theorem 3.6: Let L be an M-normal subgroup of $M = (G,H)$ and $\hat{D}$ a corepresentation of M/L. Define D on M by $D(g) = \hat{D}(gL)$ for all $g \in G$. Then

(a) D is a corepresentation of M with $L \subseteq \ker D$,

(b) D is irreducible iff $\hat{D}$ is irreducible,

(c) the intertwining numbers of D and $\hat{D}$ are equal.

In terms of characters:

Corollary 3.7: Let χ be a function on M, $\hat{\chi}$ a function on M/L, and $\chi(gL) = \hat{\chi}(g)$. Then

(a) χ is a character iff $\hat{\chi}$ is a character,

(b) χ is irreducible iff $\hat{\chi}$ is irreducible, and then they have the same intertwining number.

These three results can be used in exactly the same manner as they are in representation theory. In particular, every magnetic *single* point group is an M-homomorphic image of a magnetic *double* point group and hence the single group does not require separate treatment. While this result has been implicitly assumed by many authors we feel a proof is important as many other equally "obvious" transfers from representation theory are known to be false. In this case we may eliminate the 31 isomorphism families containing single groups from any separate calculations, to leave 33 nonisomorphic families of magnetic double point groups.

4. MAGNETIC CLASSES

An M-class C of $M = (G,H)$ was defined in N-G to be an equivalence class of elements of H: $u_1,u_2 \in C$ if there exists *either* $u \in H$ with $u\, u_1\, u^{-1} = u_2$ *or* $a \in G\text{-}H$ with $a\, u_1\, a^{-1} = u_2^{-1}$ (or both). (The term C class was used in N-G for what we here call an M-class. The prefix M is more appropriate as it is a *group* concept rather than a *corepresentation* one.) The character χ of a corepresentation is an M-class function and from this follows the column orthogonality relation for ICRs

$$\sum_i \frac{\chi_i(u_1)\chi_i(u_2)^*}{I_i} = \delta(C_{u_1},C_{u_2})\frac{|H|}{n_{u_1}},$$

where $n_u = |C_u|$ (N-G, Theorem 16).

It is well known that for ordinary groups the order of a class equals the order of the group divided by the order of the centralizer of any element of the class. A similar result holds for magnetic groups—once the centralizer is defined.

Definition 4.1: The M-centralizer $\mathbb{C}(L)$ of a set of linear operators L in M is

$$\mathbb{C}(L) = \{u,a \in M\colon ul = lu, al = l^{-1}a\ \forall l \in L\}.$$

Lemma 4.2: $\mathbb{C}(u)$ is a subgroup of M.

$\mathbb{C}(u)$ may consist of linear elements only or of both linear and antilinear elements.

Theorem 4.3: If u is an element of an M-class C of M then

$$|C|\times|\mathbb{C}(u)| = |M|.$$

This may be shown by adapting the ordinary group proof of, say, Jansen and Boon.[22] This means considering linear and antilinear elements separately and consequently is a little tedious. Similar adaptations are required in dealing with the class multiplication constants:

Definition 4.4: Let C_i and C_j be two M-classes. The class multiplication constant $h_{ij}^{\,k}$ is the number of pairs $u_i \in C_i$ and $u_j \in C_j$ whose product is any fixed element $u_k \in C_k$.

Lemma 4.5: $h_{ij}^{\,k}$ is independent of the element $u_k \in C_k$.

Proof: We prove this simple result only to demonstrate the alterations necessary for magnetic groups. Let $u_k, u_k' \in C_k$

(a) If $u_k = u u_k' u^{-1}$, then to each pair $u_i \in C_i, u_j \in C_j$ with

$u_i u_j = u_k$ corresponds another pair $u_i' = u^{-1} u_i u \in C_i$

and $u_j' = u^{-1} u_j u \in C_j$ with $u_i' u_j' = u_k'$.

(b) If $u_k = a u_k'^{-1} a^{-1}$, then to each pair $u_i \in C_i, u_j \in C_j$ with

$u_i u_j = u_k$ corresponds another pair

$u_i' = a^{-1} u_i^{-1} a \in C_i$, $u_j' = (a^{-1} u_i) u_j^{-1} (a^{-1} u_i)^{-1} \in C_j$

with

$u_i' u_j' = u_k'$.

Hence the number of pairs $u_i' u_j' = u_k'$ equals the number of pairs $u_i u_j = u_k$.

Similarly,

Theorem 4.6: With $C_1 = \{e\}$ and $C_{-i} = (C_i)^{-1}$

(a) $h_{ij}^{\,1} = |C_i| \delta_{-i,j}$,

(b) $h_{ij}^{\,k} = h_{ji}^{\,k} = h_{i-j}^{\,-k}$,

(c) if D is irreducible and $S_u = \sum_{u' \in C_u} D(u')$ then

$$S_u = \frac{|C_u| \chi(u) I}{f},$$

where I is the intertwining number of D and f the degree of D (This follows from Theorem 12 of N-G.),

(d) with S as in (c)

$$S_{u_i} S_{u_j} = \sum_{\text{all classes}} h_{ij}^{\,k} S_{u_k},$$

(e) $|C_{u_i}| \cdot |C_{u_j}| \chi(u_i) \chi(u_j) = f \sum_{\text{all classes}} h_{ij}^{\,k} |C_{u_k}| \chi(u_k)$,

(f) $h_{ij}^{\,k} = \dfrac{1}{|H|} \sum_{\substack{\text{all ICRs} \\ p}} \dfrac{|C_{u_i}| \cdot |C_j^u|}{f_p I_p} \chi_p(u_i) \chi_p(u_j) \chi_p(u_k)^*$,

(g) (van Zanten and de Vries[23])

$$\sum_{i,j,k} \frac{C_{u_k}}{|C_{u_i}| \cdot |C_{u_j}|} (h_{ij}^{\,k})^2 = |H| \sum_{\substack{\text{all ICRs} \\ p}} \frac{I_p}{f_p^2}.$$

5. DIRECT PRODUCT GROUPS

On the face of it, direct product groups are not particularly useful. For example, Cracknell[24] considers $m'3$ as the direct product $32 \times \bar{1}'$ and concludes that this is not profitable as the three ICRs of $m'3$ are not direct products of the four IRs of 32 and the one ICR of $\bar{1}'$. However, if we return to the group idea of direct products being formed of ordered pairs, then $32 \times \bar{1}'$ is a very odd group indeed as it contains the element $(E, \theta I)$, for example, which acts linearly in one space and antilinearly in another. Whilst it is possible that such mixed magnetic/linear groups may yet find applications, we investigate in this section a direct product which is conceptually simpler.

Definition 5.1: Let M_1 and M_2 be magnetic groups and set the magnetic group

$$M = M_1 \overset{M}{\times} M_2$$

to be their M-direct (outer) product if

(a) the linear subgroup of M is the direct product of the linear subgroups of M_1 and M_2.

(b) the antilinear coset of M is the direct product of antilinear cosets of M_1 and M_2.

Symbolically, if $M_1 = (G_1, H_1)$ and $M_2 = (G_2, H_2)$ then

$$M_1 \overset{M}{\times} M_2 = ((G_1 - H_1) \times (G_2 - H_2) \cup (H_1 \times H_2), H_1 \times H_2).$$

Some standard result for ordinary direct products do not transfer to magnetic direct products:

(a) $|M_1 \overset{M}{\times} M_2| = |M_1| \cdot |M_2| / 2$. This follows from the orders of the linear subgroups.

(b) Neither M_1 nor M_2 need be M-isomorphic to a subgroup of

$$M_1 \overset{M}{\times} M_2$$

as neither $\{M_1, e\}$ nor $\{e, M_2\}$ are subgroups. For example

$$4' \overset{M}{\times} 21' = \{(E, E), (E, C_2), (C_2, E), (C_2, C_2),$$

$$(\theta C_4^+, \theta), (\theta C_4^+, \theta C_2), (\theta C_4^-, \theta), (\theta C_4^- \ \theta C_2)\}$$

and $21'$ is not a subgroup.

(c) If C_i and C_j are M-classes of M_1 and M_2, respectively,

then $C_i \times C_j$ need not be an M-class of

$$M_1 \overset{M}{\times} M_2.$$

Again this is because of the absence of elements (a_1, e) and (e, a_2) from

$$M_1 \overset{M}{\times} M_2.$$

However, each direct product of M-classes splits into at most two M-classes of

$$M_1 \overset{M}{\times} M_2.$$

For example, in

$$31' \overset{M}{\times} 31'$$

the product $\{C_3^+, C_3^-\} \times \{C_3^+, C_3^-\}$ gives the two M-classes $\{(C_3^+, C_3^+), (C_3^-, C_3^-)\}$ and $\{(C_3^+, C_3^-), (C_3^-, C_3^+)\}$. On the other hand, many results are transferable:

(d) The M-direct product is commutative,

$$M_1 \overset{M}{\times} M_2 \cong M_2 \overset{M}{\times} M_1,$$

and associative,

$$(M_1 \overset{M}{\times} M_2) \overset{M}{\times} M_3 = M_1 \overset{M}{\times} (M_2 \overset{M}{\times} M_3).$$

(e) M_1 is naturally M-isomorphic to

$$(M_1 \overset{M}{\times} M_2)/H_2.$$

(f) M is naturally M-isomorphic to the diagonal subgroup of

$$M \overset{M}{\times} M \overset{M}{\times} \cdots \overset{M}{\times} M.$$

So as for ordinary groups the inner direct product may, if desired, be treated by descent in symmetry from the outer direct product.

(g) If d_1 and d_2 are corepresentations of M_1 and M_2, respectively, then $d = d_1 \times d_2$ is a corepresentation of

$$M = M_1 \overset{M}{\times} M_2.$$

From (c), irreducibility of d_1 and d_2 does not necessarily imply irreducibility of $d = d_1 \times d_2$ as the number of M-classes may increase. The ICRs of

$$M_1 \overset{M}{\times} M_2$$

are, however, easily obtained:

Theorem 5.2: Let d_1 and d_2 be ICRs of M_1 and M_2, respectively, and $d = d_1 \times d_2$ be a corepresentation of

$$M = M_1 \overset{M}{\times} M_2.$$

(a) If d_1 is of type (a), then d is irreducible and of the same type as d_2.

(b) If d_1 and d_2 are both of type (b) then d is reducible to four equivalent ICRs of type (a).

(c) If d_1 is of type (b) and d_2 of type (c) then d is reducible to two equivalent ICRs of type (c).

(d) If d_1 and d_2 are both of type (c) then d is reducible to two inequivalent ICRs D_1 and D_2 which have the same degree and are both of type (c). Further, let C_{ij} be the product of M-classes $C_i \times C_j$ of M_1 and M_2, respectively. If C_{ij} is an M-class of M, then D_1 and D_2 have equal characters on C_{ij}. If, however, C_{ij} reduces to two classes C and C', the character of $D_1(D_2)$ on C equals the character of $D_2(D_1)$ on C'.

Proof: A character based proof is possible but not particularly useful for finding the ICRs for

$$M_1 \overset{M}{\times} M_2,$$

as in general a transformation is required to reduce d. Consequently we give a constructive proof based on the definite matrix forms given in the Appendix of N–G. Most of the ICRs so far given in the literature are of this form or differing by a simple transformation.

(a) This is irreducible so no transformation is required.

(b) Let $d_i(u_i) = \begin{pmatrix} \Delta_i(u_i) & 0 \\ 0 & \Delta_i(u_i) \end{pmatrix}$ and

$$d_i(a_0^i) = \begin{pmatrix} 0 & P_i \\ -P_i & 0 \end{pmatrix}$$

with Δ_i an IR of H_i. Then d is equivalent to $d' = 4D$, where D is the ICR of type (a),

$$D((u_1, u_2)) = \Delta_1(u_1) \times \Delta_2(u_2)$$

and

$$D((a_0^1, a_0^2)) = P_1 \times P_2.$$

(c) Let $d_1(u_1) = \begin{pmatrix} \Delta_1(u_1) & 0 \\ 0 & \Delta_1(u_1) \end{pmatrix}$, $d_1(a_0^1) = \begin{pmatrix} 0 & P_1 \\ -P_1 & 0 \end{pmatrix}$

and

$$d_2(u_2) = \begin{pmatrix} \Delta_2(u_2) & 0 \\ 0 & \Delta_3(u_2) \end{pmatrix}, \quad d_2(a_0^2) = \begin{pmatrix} 0 & P_2 \\ P_3 & 0 \end{pmatrix}.$$

Then d is equivalent to $d' = 2D$, where D is the ICR of type (c),

$$D((u_1, u_2)) = \begin{pmatrix} \Delta_1(u_1) \times \Delta_2(u_2) & 0 \\ 0 & \Delta_1(u_1) \times \Delta_3(u_2) \end{pmatrix},$$

$$D((a_0^1, a_0^2)) = \begin{pmatrix} 0 & P_1 \times P_3 \\ -P_1 \times P_2 & 0 \end{pmatrix}.$$

(d) Let $d_i(u_i) = \begin{pmatrix} \Delta_i(u_i) & 0 \\ 0 & \Delta_i'(u_i) \end{pmatrix}$ and $d_i(a_0^i) = \begin{pmatrix} 0 & P_i \\ P_i' & 0 \end{pmatrix}$.

Then d is equivalent to $d' = D_1 \oplus D_2$, where D_1 and D_2 are the two ICRs of type (c),

$$D_1((u_1, u_2)) = \begin{pmatrix} \Delta_1(u_1) \times \Delta_2'(u_2) & 0 \\ 0 & \Delta_1'(u_1) \times \Delta_2(u_2) \end{pmatrix},$$

$$D_1((a_0^1, a_0^2)) = \begin{pmatrix} 0 & P_1 \times P_2' \\ P_1' \times P_2 & 0 \end{pmatrix}$$

and

$$D_2((u_1, u_2)) = \begin{pmatrix} \Delta_1(u_1) \times \Delta_2(u_2) & 0 \\ 0 & \Delta_1'(u_1) \times \Delta_2'(u_2) \end{pmatrix},$$

$$D_2((a_0^1, a_0^1)) = \begin{pmatrix} 0 & P_1 \times P_2 \\ P_1' \times P_2' & 0 \end{pmatrix}.$$

The second part of (d) follows by the equality of the traces of $D_1((u_1, u_2))$ and $D_2((u_1, au_2^{-1}a^{-1}))$.

We still have to show that this gives *all* ICRs of

$$M_1 \overset{M}{\times} M_2.$$

Firstly, if an ICR D is contained in both $d_1 \times d_2$ and $d_3 \times d_4$

then $d_1 = d_3$ and $d_2 = d_4$ by nonequivalence of characters in M_1 and M_2. Secondly, Theorem 14 of N-G related the degrees and intertwining numbers of all ICRs of a magnetic group to its order, and by calculating degrees and intertwining numbers of the ICRs of

$$M_1 \overset{M}{\times} M_2$$

obtained from those of M_1 and M_2,

Theorem 5.3: Each ICR of

$$M = M_1 \overset{M}{\times} M_2$$

is a component of $d_1 \times d_2$ for some ICRs d_1 and d_2 of M_1 and M_2, respectively.

Restating all this in terms of characters

Corollary 5.4: Let ψ_1 and ψ_2 be irreducible characters of M_1 and M_2 with intertwining numbers I_1 and I_2, respectively, and let $\chi = \psi_1\psi_2$ be a character of

$$M = M_1 \overset{M}{\times} M_2 .$$

(a) If $I_1 = 1$ then χ is irreducible with intertwining number $I = I_2$. (Of course, the subscripts "one" and "two" may be interchanged throughout).

(b) If $I_1 = I_2 = 4$ then $\chi' = \chi/4$ is irreducible with intertwining number one.

(c) If $I_1 = 4$ and $I_2 = 2$ then $\chi' = \chi/2$ is irreducible with intertwining number two.

(d) If $I_1 = I_2 = 2$ then $\chi = \chi' + \chi''$, where χ' and χ'' are both irreducible of the same degree with intertwining number two. Further,
$\chi'((u_1, u_2)) = \chi''((u_1, au_2^{-1}a^{-1}))$.

Example: *4′ has three ICRs A, DE, and DB with intertwining numbers one, two, and four, respectively (the character table appears in Table II). The M-classes of

$$*4' \overset{M}{\times} *4'$$

are $C_1 = \{(E, E)\}$, $C_2 = \{(C_{2z}, E), (\bar{C}_{2z}, E)\}$, $C_3 = \{(E, C_{2z}), (E, \bar{C}_{2z})\}$, $C_4 = \{(\bar{E}, E)\}$, $C_5 = \{(E, \bar{E})\}$, $C_6 = \{(C_{2z}, \bar{E}), (C_{2z}, \bar{E})\}$, $C_7 = \{(\bar{E}, C_{2z}), (\bar{E}, \bar{C}_{2z})\}$, $C_8 = \{(C_{2z}, C_{2z})\}$, $C_9 = \{(C_{2z}, \bar{C}_{2z}), (\bar{C}_{2z}, C_{2z})\}$, and $C_{10} = \{(\bar{E}, \bar{E})\}$. Only one direct product of M- classes splits, to $C_8 \oplus C_9$. $A \times A, A \times D\bar{E}, A \times DB, D\bar{E} \times A$, and $DB \times A$ are all irreducible. $(DB \times DB)/4$ is irreducible with $I = 1$, and $(DE \times DB)/2$ and $(DB \times DE)/2$ are both irreducible with $I = 2$. $D\bar{E} \times D\bar{E} = D_1 \oplus D_2$, both irreducible with $I = 2$. They have characters χ_1 and χ_2 (respectively) equal on C_1 through C_7, and C_{10}. $\chi_1(C_8) = \chi_2(C_9) = a$ and $\chi_1(C_9) = \chi_2(C_8) = b$. Since the character of $D\bar{E}$ on C_{2z} is zero, $a = -b$. Row or column orthogonality fixes $|a| = 2$. To determine the argument of a, additional information appears to be necessary. For example, C_8 and C_9 are ambivalent and so a is real (see next section). Alternatively, $a^2 = 4$ follows by the class multiplication rule [Theorem 4.6, part (e)].

The major problem with direct product groups is that of *identifying* when a group is a direct product. One case is always easy to spot, though: a group containing the inversion group $\bar{1} = \{E, I\}$.

Corollary 5.5: Let M contain the linear subgroup $\bar{1}$. Then

$$M \cong M' \overset{M}{\times} \bar{1}1' ,$$

where $M' = (G', H') = M/\bar{1}$. To each ICR D of M' corresponds exactly two ICRs D_g and D_u of M with equal matrices on (H', E) and opposite matrices (in sign) on (H', I).

As in representation theory, such "inversion" magnetic groups may now be dealt with trivially from the "noninversion" groups. In Sec. 3 the number of families of magnetic point groups requiring separate calculations was reduced to 33. Eliminating now the inversion groups leaves only 16.

6. SPECIAL GROUPS AND ICRs

Groups with only one-dimensional ICRs have, of course, a particularly simple character theory (inner direct products, for example, are trivial). If the degree of an ICR is only one, the intertwining algebra can only be $\mathbb{R}$ and so the intertwining number must be one. If all ICRs have degree one, from Theorem 14 of N-G the number of ICRs, which is also the number of M-classes, equals the order of the linear subgroup. Every M-class consequently has only one element and the group must satisfy the relations

$$u_1u_2 = u_2u_1, \quad \forall u_1,u_2 \in H$$

and

$$au = u^{-1}a, \quad \forall u \in H,\ a \in G - H .$$

Conversely, this guarantees that ICRs have degree one. While H is abelian, G is in general nonabelian [for example, $M = (D_n, C_n)$ for all $n \geqslant 1$ satisfies the relations]. Abelian G may have two-dimensional ICRs (for example, 4′).

To calculate the number of one-dimensional ICRs for general M we need the commutator subgroup.

Definition 6.1: The M-commutator subgroup M' is the subgroup of M generated by

$$\{u_1^{-1}u_2^{-1}u_1u_2, a^{-1}u_1au_1 : u_1,u_2 \in H, a \in G - H\}.$$

Lemma 6.2: M' is an M-normal subgroup of M.

Proof:

(a) $u(u_1^{-1}u_2^{-1}u_1u_2)u^{-1} \in M'$,

(b) $u(a^{-1}u_1au_1)u^{-1} = [(au^{-1})^{-1}u_1(au^{-1})u_1] \cdot [u_1^{-1}uu_1u^{-1}] \in M'$,

(c) $a^{-1}(u_1^{-1}u_2^{-1}u_1u_2)a = [(u_1a)^{-1}u_2^{-1}(u_1a)u_2^{-1}] \cdot [(au_2^{-1})^{-1}u_2(au_2^{-1})u_2] \in M'$,

(d) $a^{-1}(a_1^{-1}ua_1u)a = [(a_1a)^{-1}u(a_1a)u^{-1}] \cdot [(au^{-1})^{-1}u(au^{-1})u] \in M'$.

The proof then follows that for ordinary groups.

Theorem 6.3: M' is the minimal normal subgroup L such that M/L possesses only one-dimensional ICRs.

Corollary 6.4: The number of one-dimensional ICRs is $\frac{1}{2}[M:M'] = |H|/|M'|$.

Recently Butler *et al.*[8,9,25–28] have developed and used a recursive method for generating $6j$ and $3jm$ tensors for

TABLE II. Selected character tables of the magnetic point groups. The group appears on the upper left with the ICR labels beneath. In the middle, the classes are listed along the top with the characters beneath. To the right are successively the intertwining number I, the Frobenius–Schur invarant c, and n, the minimal power for the occurence of each ICR in an (arbitrarily chosen) faithful ICR.

(a)

$*4'$	E	$C_{2z},\bar{C}_{2z}$	$\bar{E}$	I	c	n
A	1	1	1	1	1	0
DB	2	-2	2	4	2	2
$D\bar{E}$	2	0	-2	2	0	1

(b)

$*4'22'$	E	$\bar{E}$	$C_{2x,y},\bar{C}_{2x,y}$	$C_{2z},\bar{C}_{2z}$	I	c	n
A	1	1	1	1	1	1	0
DE	2	2	0	-2	2	2	2
B_1	1	1	-1	1	1	1	2
$\bar{E}$	2	-2	0	0	1	-1	1

(c)

$*41'$	E	$\bar{E}$	$C_{2z},\bar{C}_{2z}$	$C_{4z}^{\pm}$	$\bar{C}_{4z}^{\pm}$	I	c	n
A	1	1	1	1	1	1	1	0
B	1	1	1	-1	-1	1	1	4
DE	2	2	-2	0	0	2	0	2
$D\bar{E}_1$	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	2	0	1
$D\bar{E}_2$	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	2	0	3

(d)

$*31'$	E	$\bar{E}$	$C_3^{\pm}$	$\bar{C}_3^{\pm}$	I	c	n
A	1	1	1	1	1	1	0
DE	2	2	-1	-1	2	0	2
$D\bar{A}$	2	-2	-2	2	4	2	3
$D\bar{E}$	2	-2	1	-1	2	0	1

(e)

$*61'$	E	$\bar{E}$	$C_6^{\pm}$	$\bar{C}_6^{\pm}$	$C_3^{\pm}$	$\bar{C}_3^{\pm}$	$C_2,\bar{C}_2$	I	c	n
A	1	1	1	1	1	1	1	1	1	0
B	1	1	-1	-1	1	1	-1	1	1	6
DE_1	2	2	-1	-1	-1	-1	2	2	0	4
DE_2	2	2	1	1	-1	-1	-2	2	0	2
$D\bar{E}_1$	2	-2	0	0	-2	2	0	2	0	3
$D\bar{E}_2$	2	-2	$-\sqrt{3}$	$\sqrt{3}$	1	-1	0	2	0	1
$D\bar{E}_3$	2	-2	$\sqrt{3}$	$-\sqrt{3}$	1	-1	0	2	0	5

(f)

$*321'$	E	$\bar{E}$	$C_3^{\pm}$	$\bar{C}_3^{\pm}$	$C'_{21,2,3},\bar{C}'_{21,2,3}$	I	c	n
A_1	1	1	1	1	1	1	1	0
A_2	1	1	1	1	-1	1	1	2
E	2	2	-1	-1	0	1	1	2
$\bar{E}_1$	2	-2	1	-1	0	1	-1	1
$D\bar{E}$	2	-2	-2	2	0	2	0	3

(g)

$*231'$	E	$\bar{E}$	$C_{2x,y,z},\bar{C}_{2x,y,z}$	$C^{\pm}_{31,2,3,4}$	$\bar{C}^{\pm}_{31,2,3,4}$	I	c	n
A	1	1	1	1	1	1	1	0
DE	2	2	2	-1	-1	2	0	4
T	3	3	-1	0	0	1	1	2
$\bar{E}$	2	-2	0	1	-1	1	-1	1
$D\bar{F}$	4	-4	0	-1	1	2	0	3

groups of linear operators. When certain problems regarding the $6j$ tensor for grey groups have been resolved[29] it is likely that the method can be adapted to magnetic groups. It is one of the few which involve properties of faithful representations and in anticipation of future use.

Theorem 6.5 [Burnside–Brauer (Ref. 21)]: Let χ be a faithful character of a magnetic group $M=(G,H)$ and suppose $\chi(u)$ takes on exactly m different values for $u\in H$. Then every irreducible character of M occurs in the nth inner Kronecker power χ^n for $0\leqslant n<m$.

In the accompanying character tables a faithful ICR is given wherever possible and the minimum value of n. A faithful ICR has kernel $\{e\}$ and hence $\chi(u)\neq\chi(e)$ for $u\neq e$.

Finally, in this section, we consider real-valued characters.

Definition 6.6: An M-class C is ambivalent if for each $u\in C$ its inverse is also in C. Alternatively, if u is any arbitrary element of C, then there exists $u_1\in H$ with $u_1uu_1^{-1}=u^{-1}$ or

$a \in G - H$ with $au = ua$.

Every grey group (i.e., containing the commuting operator θ) has every M-class ambivalent.

Theorem 6.7: The number of ICRs of a magnetic group with real character equals the number of ambivalent M-classes.

Thus the grey groups have only real characters.

7. THE INTERTWINING NUMBERS

The intertwining numbers have been seen to play a crucial role in the character theory of magnetic groups. It we already know the character table then of course the intertwining numbers are already known. However, as in the last section it is of interest to see if information can be obtained purely from group or class properties, particularly if the character table has not been determined. Here is a simple result.

Theorem 7.1: The number of ICRs with intertwining numbers one or four equals the number of M-classes C_i with the following property: for any $u \in C_i$ there exists $a \in G - H$ such that $au = u^{-1}a$.

Proof: Let u_1, u_2 be arbitrary elements of C_i and $a_1 \in G - H$ with $a_1 u_1 = u^{-1} a_1$. Then u_1 and u_2 are equivalent by a linear element. To see this, suppose they are equivalent by a nonlinear element a_2:

$$u_1 = a_2 u_2^{-1} a_2^{-1}.$$

By substitution,

$$(a_1 a_2) u_2^{-1} a_2^{-1} = u_1^{-1} a_1$$

or

$$u_2 (a_1 a_2)^{-1} = (a_1 a_2)^{-1} u_1$$

so they are equivalent by a linear element. It is readily checked that $au = u^{-1}a$ is a class property independent of the choice of u. Hence any such M-class remains irreducible on restriction to ordinary classes of H. Conversely, if an M-class does not possess this property then it branches into two ordinary classes of H. But from the relations between IRs of H and ICRs of M (N-G, Appendix) the number of ICRs with intertwining number two equals the number of M-classes which split on H, and hence the number with intertwining number one or four is the number of irreducible M-classes on H as required.

This theorem completely determines the number of ICRs with $I = 2$ by M-class properties. The problem of deciding between the number with $I = 1$ and the number with $I = 4$ is much more complex. For example, for a grey group it becomes the calculation of the number of IRs of the first and second kinds, respectively. van Zanten and de Vries[30] and Gow[31] have given various lower bounds for these, but only in certain cases are there presently exact solutions.

For the remainder of this section we aim at a special case, namely when all ICRs have $I = 1$. In this case the character theory of the magnetic group reduces to that of the linear subgroup and, especially for magnetic point groups, this may be very well known. Extensions along the lines of van Zanten and de Vries[30] will be obvious.

Definition 7.2: Let $\zeta^{(2)}(u)$ be the number of square roots of u in G-H.

Lemma 7.3: $\zeta^{(2)}$ is an M-class function.

Lemma 7.4: $\zeta^{(2)} = \Sigma_i c_i \chi_i$, where $c_i = 1$ if $I_i = 1$, $c_i = -\frac{1}{2}$ if $I_i = 4$, and $c_i = 0$ if $I_i = 2$.

Proof: From N-G, row orthogonality gives

$$c_i = \frac{1}{I_i |H|} \sum_u \zeta^{(2)}(u) \chi_i(u)^*.$$

But

$$\zeta^{(2)}(u) \chi_i(u)^* = \sum_{a \in G - H : a^2 = u} \chi_i(a^2),$$

so

$$c_i = \frac{1}{I_i |H|} \sum_a \chi_i(a^2).$$

Substituting by Eqs. (20), (24), and (28) of N-G the result follows.

Theorem 7.5: All ICRs of a magnetic group have intertwining number one iff

$$\zeta^{(2)}(e) = \sum_i \chi_i(e).$$

Proof: Immediate from the possible values of c_i.

Corollary 7.6: Let the set of irreducible characters of M be ICR(M) and the set of linear irreducible characters of H be Irr(H). Then ICR(M) = Irr(H) iff

$$\zeta^{(2)}(e) = \sum_i \varphi_i(e) \quad \text{for } \varphi_i \in \text{Irr}(H).$$

Proof: If ICR(M) = Irr(H) then all ICRs of M must have intertwining numbers of one to avoid branching, and hence the previous theorem applies with $\varphi \in \text{Irr}(H)$ replacing $\psi \in \text{ICR}(M)$.

Conversely, suppose

$$\zeta^{(2)}(e) = \sum_i \varphi_i(e).$$

Break this up into a sum over j[IRs inducing ICRs of type (a)], k [IRs inducing ICRs of type (b)], and 1 [IRs inducing ICRs of type (c)]:

$$\zeta^{(2)}(e) = \sum_j \varphi_j(e) + \sum_k \varphi_k(e) + \sum_l \varphi_l(e).$$

By the relations between IRs and ICRs this is

$$\zeta^{(2)}(e) = \sum_j \chi_j(e) + \tfrac{1}{2} \sum_k \chi_k(e) + \tfrac{1}{2} \sum_l \chi_l(e).$$

But we know from Lemma 7.4 that

$$\zeta^{(2)}(e) = \sum_j \chi_j(e) - \tfrac{1}{2} \sum_k \chi_k(e).$$

and as the sums over k and l are nonnegative they must vanish. Hence all intertwining numbers are one and ICR(M) = Irr(H).

Example: The group *6′2′2 has antilinear elements $\{\theta C_6^{\pm}, \theta \bar{C}_6^{\pm}, \theta C_2, \theta \bar{C}_2, \theta C''_{21,2,3}, \theta \bar{C}''_{21,2,3}\}$ and eight of these square to the identity. The linear subgroup *32 has six IRs and the sum of their degrees is also eight. Hence *6′2′2 has the same character table as *32.

The character theory of this type of group follows from the linear group and may be found in many places.[1,5,9] Eliminating these from the remaining 16 families of magnetic groups leaves only seven families—a very manageable number! Their character tables are given in Table II.

8. COMPLEX CONJUGATES OF ICRs WITH REAL CHARACTER

As with representations, a corepresentation is equivalent to its complex conjugate iff it has real character. The row orthogonality relations of N–G give an immediate character test

$$\sum_u (\chi(u))^2 \neq 0 \quad \text{iff } D \cong D^*.$$

For linear groups the well-known Frobenius–Schur invariant[21] divides IRs with real characters into orthogonal IRs ($c = 1$) and symplectic IRs ($c = -1$). This division is of great importance for Racah algebra methods of linear groups as it completely determines the $1-j$ phase which is required for, amongst other things, permutation properties of the $6j$ tensor.[6,7] (For complex IRs the phase is undetermined. However, the concept of quasiambivalence[32] has proved useful for a partial determination of the phase.[7,33]) Newmarch and Golding[29] have similarly found the $1-j$ phase important for the Racah algebra of grey groups but have noted that for ICRs of types (b) or (c) of these groups the phase is not uniquely determined.[34] Thus the relations between complex conjugates and to the $1-j$ phase deserves further investigation.

For the remainder of this section, D will be a unitary ICR with real character, D^* the ICR with matrices complex conjugate to D (that D^* is an ICR is easily shown) and $\mathbf{p}$ the set of matrices giving equivalence of D to D^*:

$$\mathbf{p} = \{ P : PD(u) = D(u)^*P, PD(a) = D(a)^*P^* \forall u,a \in M \}.$$

$\mathbf{m}$ is the commutator algebra of D, i.e., the set of all matrices commuting with D.

These are all simple generalizations of results on representations:

Lemma 8.1:

(a) If P is any element of $\mathbf{p}$ and M any element of $\mathbf{m}$ then both PM and M^*P are elements of $\mathbf{p}$.

(b) If P, Q are two elements of $\mathbf{p}$, there exist $M,M' \in \mathbf{m}$ such that $P = QM' = M^*Q$.

(c) If $P \in \mathbf{p}$, $P^*P \in \mathbf{m}$.

While not affecting his conclusions, Rudra[35] makes an error in stating $P^*P = \lambda E$, as can be shown by example (a similar error is made by Kotzev and Aroyo[36] in connection with isoscalars). Consider the two-dimensional ICR of $4'$ generated by

$$D(\theta C_{4z}^+) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

From the reality of D, $\mathbf{p} = \mathbf{m}$ and the most general form of $P \in \mathbf{p}$ is

$$P = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}.$$

Trivially, $P^*P \neq E$ except for special cases. The meat of this section is that such special cases *must* occur.

Theorem 8.2: Let D be a unitary ICR with intertwining number I. Then there exist $P_0, P_1, \ldots, P_{I-1} \in \mathbf{p}$ such that

(a) $P_i = P_0 M_i$, where the M_i form a group of

$$\mathbf{m}\{ \pm E, \pm M_1, \pm M_2, \ldots \},$$

(b) $P_i^* P_i = c_i E$ with $c_i^2 = 1$, $i = 0,1,\ldots,I-1$,

(c) if $I = 4$, $c_0 c_1 c_2 c_3 = -1$.

Proof: It is sufficient to take D of type (b) with intertwining number four as the other two types follow as special cases. Choose any unitary $P \in \mathbf{p}$ and set $M = P^*P$. M is also unitary and may be written

$$M = x_1 E + x_2 M_1$$

with x_1, x_2 real, $x_1^2 + x_2^2 = 1$, and $M_1 \in \mathbf{m}' = \mathbf{m} - \{\lambda E : \lambda \in \mathbb{R}\}$ with $M_1^2 = -E$. Suppose $x_2 \neq 0$. Then M possesses an inverse square root

$$M^{-1/2} = \frac{\sqrt{1+x_1}}{\sqrt{2}} E - \frac{\sqrt{1-x_1}}{\sqrt{2}} M_1.$$

From $P^*P = M$ and $PP^* = M^*$, $PM = M^*P$ and so $PM_1 = M_1^*P$. Hence

$$PM^{-1/2} = M^{-(1/2)*}P.$$

If now we set

$$P_0 = PM^{-1/2} \text{ and } P_1 = PM^{-1/2}M_1$$

it follows that

$$P_0^* P_0 = E \text{ and } P_1^* P_1 = -E.$$

Continuing with this case of $x_2 \neq 0$, by a suitable $4-D$ rotation in $\mathbf{m}$ M_1 can be taken as an element of the group of $\mathbf{m} : \{ \pm E, \pm M_1, \pm M_2, \pm M_1 M_2\}$ with M_2 arbitrarily lying in a plane orthogonal to E and M_1. Set $P_2 = P_0 M_2$ and define $M' \in \mathbf{m}$ by

$$M' = P_2^* P_2.$$

A simple equation relates M' and M_1. Consider

$$\begin{aligned}(P_0 M_1 M_2)^* (P_0 M_1 M_2) &= (P_0^* M_1^* P_0)(P_0^* M_2^* P_0) M_1 M_2 \\ &\text{as } P_0 P_0^* = E \\ &= -M_1^{-1} M' M_2^{-1} M_1 M_2 \\ &\text{as } P_1^* P_1 = P_0^* M_1^* P_0 M_1 = -E \\ &\text{and } P_0^* M_2^* P_0 = P_2^* P_2 M_2^{-1} = M' M_2^{-1} \\ &= -M_1 M' M_1 \text{ as } M_1 M_2 = -M_2 M_1 \text{ and } M_2^2 = -E.\end{aligned}$$

But this also equals

$$(P_0 M_2 M_1)^* (P_0 M_2 M_1) = M'$$

in a similar manner. Writing M' as a linear combination of M_1, M_2, and $M_1 M_2$ and equating these gives

$$M' = y_1 E + y_2 M_1.$$

By unitarity of all matrices, $y_1^2 + y_2^2 = 1$. If in this equation $y_2 = 0$ with $y_1 = \pm 1$, set

$$P_3 = P_0 M_1 M_2.$$

Then $P_2^* P_2 = P_3^* P_3 = y_1 E$ and indeed, for all real linear combinations $P' = z_1 P_2 + z_2 P_3$ with $z_1^2 + z_2^2 = 1$,

$$P'^* P' = y_1 E.$$

If, however, $y_2 \neq 0$, a contradiction rapidly follows. For then taking the inverse square root of M' as with M, a real linear combination P_2' of P_2 and P_3 exists with $P_2'^* P_2' = yE$ and this P_2' may be used in place of P_2. From this it follows that y_2 must have been zero after all.

The case excluded so far was of $x_2 = 0$. However, either for all $P' \in \mathbf{p}$ the corresponding x_2' is zero, in which case there is nothing to show, or there is at least one for which $x_2' \neq 0$ and this P' may be used in place of P in the above analysis.

For an ICR of type (a), $\mathbf{m} \cong \mathbb{R}$ and there is nothing to show. For an ICR of type (b) $m \cong \mathbb{C}$ and either all $P \in \mathbf{p}$ satisfy $P^*P = cE$ or P_0 and P_1 can be constructed as above.

Part (c) follows from the equation

$$c_0 P_3^* P_3 = -P_0^* M_1^* P_0 P_0^* M_2^* P_0 M_2 M_1$$

to complete the proof.

Character tests may be established in a fairly straightforward manner from the orthogonality relations for ICRs given in N-G Sec. 3.

Theorem 8.3: Let D be a unitary ICR equivalent to D^*, and let P_i, c_i be as in the preceding theorem. Then

$$\frac{1}{|H|} \sum_u \chi(u^2) = \sum_i c_i.$$

In conjunction with $c_i^2 = 1$ and $c_0 c_1 c_2 c_3 = -1$ for ICRs of type (b), this shows that the c_i are essentially determined by character theory alone, independent of any specific choice of P_i or of the basis for D. As usual, $c_i = 1$ means P_i is symmetric, $c_i = -1$ means P_i is antisymmetric. Setting the Frobenius–Schur invariant to be

$$c = \sum c_i$$

gives

Corollary 8.4: Let D be a unitary ICR equivalent to D^*.

(a) If D is of type (a), $c = c_0 = \pm 1$.

(b) If D is of type (b), then $c = \pm 2$. If $c = 2$, three of the c_i are positive and one negative, whereas if $c = -2$, three of the c_i are negative, one positive.

(c) If D is of type (c), $c = 2, 0, -2$. If $c = 2$, $c_0 = c_1 = 1$, if $c = -2$, $c_0 = c_1 = -1$, and if $c = 0$, c_0 and c_1 are of opposite sign.

It can be seen that for all type (b) and some type (c) ICRs there is a freedom in the choice of $1-j$ phase which does not exist for linear groups. For quasiambivalent linear groups a useful simplification for Racah methods is that the product of three $1-j$ phases is one whenever the triple product of IRs contains the identity IR.[7,8,32,33] By considering, for example, *31′ and *4′ it can be verified that the product of phases is not unity for all choices in magnetic groups even when the character is real. Whilst we do not wish to pursue this here in any depth, we do note a special case of particular relevance to grey groups: if θ is some antilinear element of a magnetic group M which commutes with all elements of the group and for which $D(\theta^2) = \pm E$ for all ICRs of M, then as in Newmarch and Golding,[20] $D(\theta) \in \mathbf{p}$ for each ICR. As

$$D(\theta^2) = D(\theta) D(\theta)^*,$$

$$(D_1(\theta) \otimes D_2(\theta))^* (D_1(\theta) \otimes D_2(\theta)) = D_1(\theta^2)^* \otimes D_2(\theta^2)^* = \pm E.$$

Hence for any D_3 in this direct product, $c_1 c_2 c_3 = 1$.

Another well-known property of the Frobenius–Schur invariant is its relation to the multiplicity of the identity IR in the symmetrized and antisymmetrized Kronecker squares $D^{[2]}$ and $D^{[1^2]}$ of an IR. From Eq. (20), (23), and (27) and Sec. 5 of N-G the Frobenius–Schur invariant for magnetic groups similarly characterizes these multiplicities for ICRs with real character. The results are summarized in Table III, from which it may be observed that the occurrence of the identity ICR in the symmetrized (antisymmetrized) Kronecker square equals the number of c_i with value one (minus one).

Finally, a word about matrix forms. If $P \in \mathbf{p}$ with $P^*P = E$ is symmetric then exactly as for linear groups, D is equivalent to a real ICR.[3] Similarly, if $P \in \mathbf{p}$ with $P^*P = -E$ is antisymmetric, then D is equivalent to a symplectic ICR.[4,34,37] Any type (b) and some type (c) ICRs (with $c = 0$) with real character are equivalent to both real and symplectic ICRs. For example, consider the ICR of type (c) with $c = 0$, DE of 41′. Constructing the ICR in the usual way from the linear subgroup gives

$$D(C_{4z}^-) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad D(\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A symmetric $P \in \mathbf{p}$ is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = r^{-1} \omega r.$$

Transforming D by $\omega_1 r$, where $\omega_1^2 = \omega$, gives

$$D'(C_{4z}^-) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad D'(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is real. On the other hand, transforming D by λE with $\lambda^2 = i$ gives

$$D''(C_{4z}^-) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad D''(\theta) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

which is in symplectic form. These provide alternative

TABLE III. The multiplicity of the identity ICR 1 in symmetrized and antisymmetrized Kronecker squares of ICRs with real character.

Type of ICR	Frobenius–Schur invariant	Multiplicity of 1 in $D^{[2]}$	Multiplicity of 1 in $D^{[1^2]}$
(a)	1	1	0
	−1	0	1
(b)	2	3	1
	−2	1	3
(c)	2	2	0
	0	1	1
	−2	0	2

"standard" forms to the one obtained by induction from the linear subgroup.

9. CONCLUSION

In a single paper it is, of course, impossible to consider all aspects of character theory used for linear groups and we have singled out a few of general interest. They should be sufficient, however, to show that character theory is a viable tool for the examination of magnetic groups.

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