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The character table for the corepresentations of magnetic groups

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A square character table is shown to exist for all finite magnetic groups. The table possesses row and column orthogonality properties similar to the character table for linear groups.

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1. INTRODUCTION

In dealing with the problem of time reversal symmetry in a group theoretic way, Wigner[1] introduced the concept of a corepresentation of a group G of linear/antilinear operators analogous to a representation of a group of linear operators only. It was soon realized that this theory had a ready physical application in dealing with magnetic crystals, where both linear and antilinear operators commute with the Hamiltonian.[2] It is also likely that this theory can be applied to the study of elementary particles due to T or CPT invariance.[3]

Despite its usefulness though, the theory of corepresentations has unpleasant features as many results from the representation theory of groups over the complex numbers do not appear to hold. Let G be a group of linear/antilinear operators, and H the subgroup of linear operators. A corepresentation D of G is a set of matrices over the complex numbers

$$D = \{ D(u), D(a) : u \in H, a \in G - H \}$$

satisfying the following rules

$$\begin{aligned} D(u_1 u_2) &= D(u_1) D(u_2), \\ D(ua) &= D(u) D(a), \\ D(au) &= D(a) D(u)^*, \\ D(a_1 a_2) &= D(a_1) D(a_2)^*, \end{aligned} \tag{1}$$

where the asterisk denotes complex conjunction. Then

(a) if M is a matrix commuting with D in the sense

$$MD(u) = D(u)M \text{ and } MD(a) = D(a)M^*.$$

Then M is a scalar matrix if and only if it has a *real* eigenvalue.[4]

(b) if D is irreducible,

$$\sum_u D(u)_{ij} D(u)^*_{lk} + \sum_a D(a)_{ik} D(a)^*_{lj} = \frac{|G|}{f} \delta_{il} \delta_{jk},$$

where $|G|$ is the order of G and f the dimension of D.[5] Note how j and k are interchanged in the two sums.

(c) the character of the matrix of an antilinear operator is not invariant under a change of basis. This follows from the transformation rule[6]

$$D'(a) = P^{-1} D(a) P^*. \tag{2}$$

(d) the number of classes need not equal the number of irreducible corepresentations (ICR's). This and the next result can be verified from Cracknell[7] or Newmarch and Golding.[8]

(e) the sum of the squares of the dimensions of the ICR's need not equal the order of G.

After deriving (a) and (b) Dimmock[5] commented ".... further development of the representation theory of nonunitary groups (without using the representation theory of the linear subgroup) has so far proven untenable." He, and others following him, have then relied heavily on the representation theory of linear groups to obtain results about corepresentations (we are not excepted from this!). In particular, the reduction of direct products is usually performed through the intermediary of the irreducible representations of the linear subgroup.[6]

This inevitably gives the impression that corepresentation theory is a poor 'second cousin' to representation theory. In a recent book Cracknell[9] is forced to defend the use of corepresentation theory for magnetic materials against those who feel that ordinary representation theory is quite sufficient, and moreover, has better properties. The best theoretical argument against this view is a demonstration that all fundamental results in representation theory are mirrored by similar fundamental results in corepresentation theory, proved *without using any theorems on representations*. In this paper it is demonstrated that, with certain generalizations and additional concepts, a square character table exists for a finite magnetic group and that this table posseses row and column orthogonality.

All of the results contained here can in fact be derived in a simpler manner by use of representation theory (cf. the character test for the types of ICR). We do not adopt that course as we wish to show that corepresentation can stand independently of representation theory.

First, some preliminary results. From Eq. (1)

$$D(u)^{-1} = D(u^{-1}) \text{ and } D(a)^{-1} = D(a^{-1})^*.$$

Definition: Two corepresentations D_1 and D_2 are *equivalent* if there exists a matrix M such that

$$MD_1(u) = D_2(u)M \text{ and } MD_1(a) = D_2(a)M^*$$

for all $u, a \in G$. The matrix M is said to *intertwine* D_1 and D_2. If D_1 equals D_2, M *commutes* with D_1.

Theorem 1: Every corepresentation is equivalent to a corepresentation by unitary matrices. This has been shown by Dimmock.[5]

Definition: A corepresentation is reducible if it is equivalent to a corepresentation of the form

$$\begin{pmatrix} D_1 & D_2 \\ 0 & D_3 \end{pmatrix}.$$

Otherwise it is irreducible an (ICR).

Theorem 2 (Mashke): Every corepresentation is equivalent to a direct sum of irreducible corepresentations. This has been given before.[6]

2. SCHUR'S LEMMAS

An algebraist once remarked to us "but nothing *interesting* happens in ordinary representation theory!" To some extent we can now sympathize with this view, as what is lost in simplicity is here compensated for by variety, with four useful forms of Schur's lemmas.

Theorem 3 (Schur I): A matrix M intertwining two ICR's D_1 and D_2 is either nonsingular or is zero.

Theorem 4 (Schur II): If M is Hermitian and commutes with a unitary ICR D then M is a real constant matrix. Both of these have been shown by Dimmock.[5]

Theorem 5 (Schur III): If D is a unitary ICR, and M a matrix satisfying $MD(u) = D(u)M$ and $M^+D(a) = D(a)M^*$ for all u, $a \in G$ then M is a constant matrix.

Proof: From

$$\begin{aligned} D(a_1a_2)M &= MD(a_1a_2), \\ D(a_1)D(a_2)^*M &= MD(a_1)D(a_2)^*, \\ \text{or } D(a_1)M^{+*}D(a_2)^* &= MD(a_1)D(a_2)^*. \end{aligned}$$

Hence

$$D(a_1)M^{+*} = MD(a_1).$$

Similarly, from

$$\begin{aligned} D(ua)M^* &= M^+D(ua), \\ D(u)M^+ &= M^+D(u). \end{aligned}$$

Together with the assumptions

$$D(u)(M+M^+) = (M+M^+)D(u)$$

and

$$D(a)(M+M^+)^* = (M+M^+)D(a)$$

for all u, $a \in G$. By Schur II,

$$M + M^+ = \lambda I.$$

Next, from the linearity of u and antilinearity of a,

$$\begin{aligned} D(u)(iM) &= iMD(u), \\ D(a)(iM)^* &= -iM^+D(a), \\ D(u)(iM^+)^* &= iM^+D(u), \\ D(a)(iM^+)^* &= -iMD(a). \end{aligned}$$

So $iM - iM^+$ also satisfies Schur II and is a constant matrix. Hence M is constant as required.

The restriction imposed on M in Schur II that it be Hermitian is a very real one. If it is not, we have already shown[8] that M is nonconstant. It is not possible to say much about any single such matrix, but we can derive results about the *set* of commuting matrices:

$$\mathbf{m} = \{M{:}M \text{ commutes with } D\},$$

$\mathbf{m}$ is closed under matrix multiplication and addition; if $M \in \mathbf{m}$ then so is M^{-1}; it is also closed under scalar multiplication by $\mathbb{R}$, and finally if $M \neq 0$, $kM \neq 0$ for any integer k.

Hence $\mathbf{m}$ is a (skew) field of characteristic zero over $\mathbb{R}$. We can say more about $\mathbf{m}$. Any $M \in \mathbf{m}$ can be written as the sum of a Hermitian and a skew-Hermitian matrix, and it is simple to show that both these belong to $\mathbf{m}$. By Schur II, $\mathbf{m}$ can thus be written as a direct sum

$$\mathbf{m} = \{\lambda I{:}\lambda \in \mathbb{R}\} \oplus \mathbf{m}', \tag{3}$$

where $\mathbf{m}'$ contains only skew-Hermitian matrices. For any $M \in \mathbf{m}'$, M^2 is Hermitian, and since its eigenvalues are negative,

$$M^2 = -\mu^2 I, \quad \text{with } \mu \text{ real.} \tag{4}$$

As $\mathbf{m}'$ is closed under multiplication by $\mathbb{R}$, it follows that for $\mathbf{m}'$ nonempty we can find elements $M_1, M_2, M_3, \cdots$ such that

$$M_i^{-1} = M_i^+ = -M_i. \tag{5}$$

With these preliminaries out of the way, we now show

Theorem 6 (Schur IV): $\mathbf{m}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$.

Proof: If $\mathbf{m}'$ is empty, then by Schur II $\mathbf{m}$ is isomorphic to $\mathbb{R}$. Assume, then, that $\mathbf{m}'$ is nonempty. The proof is in two parts: First it is shown that $\mathbf{m}$ contains a multiplicative subgroup isomorphic to the group of $\mathbb{C}$ or the group of $\mathbb{Q}$. Then, it is shown that the algebra of this group over $\mathbb{R}$ equals $\mathbf{m}$.

Let G be a multiplicative subgroup of $\mathbf{m}$ consisting of elements

$$\begin{aligned} G = \{ &\pm I, \pm M_1, \ \pm M_2, \cdots{:}M_i \in \mathbf{m}', \\ &M_i^2 = -I, M_iM_j = -M_jM_i \text{ for all } i,j \neq i\}. \end{aligned}$$

As $M_iM_j = -M_jM_i, M_j \neq M_i$ for $i \neq j$. If such a subgroup only contains the four elements

$$\pm I, \pm M_1,$$

then it is isomorphic to the group of $\mathbb{C}$.

Suppose then it contains more. It cannot contain only six elements for this would mean that M_1M_2 is a multiple of I, M_1, or M_2, which gives a contradiction. Thus it will contain at least eight, and we show that this is the maximum. For consider any $M \in G$ which is not a multiple of I, M_1 or M_2. Then

$$M_1M_2M$$

is Hermitian as $M_i^+ = -M_i$ and all M_i anticommute. Hence by Schur II

$$M_1M_2M = \lambda I \quad \text{with } \lambda \text{ real.}$$

As $M_i^2 = -I$, $\lambda = \pm 1$, so

$$M = \pm M_1M_2.$$

Therefore

$$G = \{\pm I, \pm M_1, \pm M_2, \pm M_1M_2\},$$

which is easily seen to be the quaternion group. Thus G is either the group of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$. It is not hard to check that this is a property of $\mathbf{m}$ rather than the particular group, i.e., if one group is isomorphic to Q, then all are etc. and we can refer to *the* group of $\mathbf{m}$.

For the second part of the proof, we consider the case when the group of $\mathbf{m}$ is the quaternion group as the other two follow as special cases. Further, to show that any matrix in $\mathbf{m}$ belongs to the algebra over $\mathbb{R}$ of G, it is sufficient to show that any $M \in \mathbf{m}'$ is a real linear combination of M_1, M_2, and M_1M_2.

By Hermiticity and Schur II,

$$MM_1 + M_1M = aI,$$
$$MM_2 + M_2M = bI,$$
$$MM_1M_2 + M_1M_2M = cI,$$

with a, b, c real. Set

$$N = 2M + aM_1 + bM_2 + cM_1M_2.$$

Clearly $N \in \mathbf{m}'$.

It follows that N anticommutes with M_1, M_2, and M_1M_2.

Hence N is either zero or by normalization an element of the group of $\mathbf{m}$. As there are no other elements of this group, N equals zero and

$$M = -\tfrac{1}{2}(aM_1 + bM_2 + cM_1M_2) \tag{6}$$

as required.

Thus there are possibly three kinds of ICR according as $\mathbf{m}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$. That these three types actually occur is shown by our earlier work.[8] It is helpful to quantize this by introducing the intertwining number from the pure mathematicians' version of group theory.[10] Recall that any complex number may be written as an ordered pair of real numbers, and that any quaternion may be written as an ordered quadruple of real numbers. This leads to the following.

Definition: The intertwining number I of $\mathbf{m}$ is the dimension of $\mathbf{m}$ as an algebra over $\mathbb{R}$. An ICR is of type (a) if $\mathbf{m}$ is isomorphic to $\mathbb{R}$ in which case $I = 1$; of type (b) if $\mathbf{m}$ is isomorphic to $\mathbb{Q}$, when $I = 4$, and of type (c) if $\mathbf{m}$ is isomorphic to $\mathbb{C}$ with $I = 2$.

3. ORTHOGONALITY RELATIONS

The general forms of the orthogonality relations have previously been given by Dimmock.[5] They are

Theorem 7: If D_1 and D_2 are two inequivalent ICR's,

$$\sum_u D_1(u)_{ij} D_2(u)^*_{lk} = \sum_a D_1(a)_{ik} D_2(a)^*_{lj} = 0. \tag{7}$$

For D irreducible and unitary,

$$\sum_u D(u)_{ij} D(u)^*_{lk} + \sum_a D(a)_{ik} D(a)^*_{lj} = \delta_{il}\delta_{kj}|G|/f, \tag{8}$$

where f is the dimension of D.

We only remark that the last part of this theorem may be shown in a simpler manner as

$$M = \sum_u D(u) X D(u^{-1}) + \sum_a D(a) X^T D(a^{-1})^*$$

satisfies the conditions of Schur III and hence is diagonal.

This theorem does not take into account the different types of ICR and their properties. The following is proved for an ICR of type (b) and is specialized to types (a) and (c) later.

Theorem 8: If D is a unitary ICR of type (b) with the group of $\mathbf{m}$ generated by M_1 and M_2 then

$$\sum_u D(u)_{ij} D(u)^*_{lk}$$
$$= (|G|/2f)\delta_{il}\delta_{jk} - (|G|/2f)(M_1)_{kj}(M_1)_{il}$$
$$- (|G|/2f)(M_2)_{kj}(M_2)_{il} - (|G|/2f)(M_1M_2)_{kj}(M_1M_2)_{il}. \tag{9}$$

Proof: From Schur IV, the following matrix is in $\mathbf{m}$ and can be written

$$\sum_u D(u) X D(u)^+ + \sum_a D(a) X^* D(a)^+$$
$$= \lambda I + \mu M_1 + \omega M_2 + \delta M_1M_2 \tag{10}$$

with λ, μ, ω, δ real. Taking Hermitian adjoints

$$\sum_u D(u) X^+ D(u)^+ + \sum_a D(a) X^T D(a)^+$$
$$= \lambda I - \mu M_1 - \omega M_2 - \delta M_1M_2. \tag{11}$$

Adding and taking traces,

$$(|G|/2f)(\mathrm{tr}X + \mathrm{tr}X^*) = \lambda. \tag{12}$$

By pre- and post-multiplying these by M_1 we can isolate the term μI to give

$$\mu = -(|G|/2f)[\mathrm{tr}(XM_1) + \mathrm{tr}(XM_1)^*] \tag{13}$$

with similarly

$$\omega = -(|G|/2f)[\mathrm{tr}(XM_2) + \mathrm{tr}(XM_2)^*] \tag{14}$$

and $\delta = -(|G|/2f)[\mathrm{tr}(XM_1M_2) + \mathrm{tr}(XM_1M_2)^*]$. (15)

From Schur III we also have

$$\sum_u D(u) X D(u)^+ + \sum_a D(a) X^T D(a)^+ = zI, \tag{16}$$

with Hermitian adjoint

$$\sum_u D(u) X^+ D(u)^+ + \sum_a D(a) X^* D(a)^+ = z^* I, \tag{17}$$

where $z = (G/f)\mathrm{tr}X$. (18)

The sum over a may be eliminated from Eqs. (10) and (17) to give

$$\sum_u D(u)(X - X^+) D(u)^+$$
$$= (\lambda - z^*)I + \mu M_1 + \omega M_2 + \delta M_1M_2.$$

By setting $X_{jk} = 1$ for some j, k and zero otherwise, and then setting $X_{jk} = i$ for the same j, k and zero otherwise, simple manipulations give the result.

These may be specialized to a type (c) ICR by setting $M_2 = 0$ and to a type (a) ICR by also setting $M_1 = 0$. We give a summary for each case, together with the character tests which follow directly with Eq. (8).

Type (a):

$$\sum_u D(u)_{ij} D(u)^*_{lk} = \sum_a D(a)_{ik} D(a)^*_{lj} = \frac{|G|}{2f}\delta_{il}\delta_{jk}, \tag{19}$$

$$\sum_u \chi(u)\chi(u)^* = \sum_a \chi(a^2) = \frac{|G|}{2}. \tag{20}$$

Type (b):

$$\sum_u D(u)_{ij} D(u)^*_{lk} = \frac{|G|}{2f}\delta_{il}\delta_{jk} - \frac{|G|}{2f}(M_1)_{kj}(M_1)_{il}$$
$$- \frac{|G|}{2f}(M_2)_{kj}(M_2)_{il}$$
$$- \frac{|G|}{2f}(M_1M_2)_{kj}(M_1M_2)_{il}, \tag{21}$$

$$\sum_a D(a)_{ik} D(a)^*_{lj} = \frac{|G|}{2f}\delta_{il}\delta_{jk} + \frac{|G|}{2f}(M_1)_{kj}(M_1)_{il} + \frac{|G|}{2f}(M_2)_{kj}(M_2)_{il} + \frac{|G|}{2f}(M_1M_2)_{kj}(M_1M_2)_{il}, \quad (22)$$

$$\sum_u \chi(u)\chi(u)^* = 2|G|, \quad (23)$$

$$\sum_a \chi(a^2) = -|G|. \quad (24)$$

Type (c):

$$\sum_u D(u)_{ij} D(u)^*_{lk} = \frac{|G|}{2f}\delta_{il}\delta_{jk} - \frac{|G|}{2f}(M_1)_{kj}(M_1)_{il}, \quad (25)$$

$$\sum_a D(a)_{ik} D(a)^*_{lj} = \frac{|G|}{2f}\delta_{il}\delta_{jk} + \frac{|G|}{2f}(M_1)_{kj}(M_1)_{il}, \quad (26)$$

$$\sum_u \chi(u)\chi(u)^* = |G|, \quad (27)$$

$$\sum_a \chi(a^2) = 0. \quad (28)$$

Equations (7), (20), (23), and (27), when combined with the intertwining number, are actually the row orthogonality relations of the character table. We defer the statement for a discussion of the class concept.

4. CLASSES IN COREPRESENTATION THEORY

It has already been remarked that the number of classes need not equal the number of ICR's (it is always equal to or larger). An examination of previously published tables[7] also shows that in many cases different classes have the same character for all ICR's. Clearly then, the definition of class must be extended for corepresentation theory.

Definition: Two elements u_1 and u_2 of the linear subgroup H are said to be in the same corepresentation class (C class) if *either* $u_1 = u\, u_2 u^{-1}$ for some $u \in H$ *or* $u_1 = a u_2^{-1} a^{-1}$ for some $a \in G - H$.

It is straightforward to check that this is an equivalence relation on H so that a C class may be labelled C_u where u is any element of the C class. This also follows easily:

Theorem 9: The character of a corepresentation is a C-class function on H.

The C class is here only defined over the linear subgroup H; it does not as yet appear useful to extend it to $G - H$.

Theorem 10 (Row Orthogonality): If D_i and D_j are two unitary ICR's with characters on H of χ_i and χ_j respectively, and the number of elements in C_u is n_u, then

$$\sum_{C_u} n_u \chi_i(u)\chi_j(u)^* = \delta_{ij} I_i |H|,$$

where the sum is over all C classes of H and I_i is the intertwining number of D_i.

This follows as stated at the end of the last section. Immediate results from this are

Corollary 1: If D is a corepresentation equivalent to a direct sum of ICR's D_i

$$D = \oplus \sum_i d_i D_i,$$

then

$$d_i = \frac{1}{I_i |H|} \sum_{C_u} n_u \chi(u)\chi_i(u)^*.$$

Corollary 2: If two corepresentations have the same character on H, they are equivalent. Returning to results on C classes,

Theorem 11: (a) $u' C_u u'^{-1} = C_u$ and $a C_u a^{-1} = C_u^{-1}$ for all $u, a \in G$. (b) C_u and C_u^{-1} are in one-to-one correspondence under the mapping $g \to g^{-1}$.

Theorem 12: Let D be a unitary ICR and

$$S_u = \sum_{u' \in C_u} D(u').$$

Then $S_u = zI$.

Proof: This follows by using the previous theorem to show that S_u satisfies the conditions of Schur III.

5. THE REGULAR COREPRESENTATION

The regular corepresentation D_R is useful in corepresentations for exactly the same reasons as the regular representation is; with the elements of G ordered in some arbitrary fixed order, define

$$D_R(g)_{ij} = 1 \text{ if } g = g_i g_j^{-1}$$
$$= 0 \text{ if } g \neq g_i g_j^{-1}.$$

Due to the reality of the matrices, this representation is also a corepresentation. The following are shown in exactly the same manner as in representation theory[11]—once the basic C-class results and row orthogonality are known, the methods of the two theories coincide.

Theorem 13: The number of times an ICR D_i is contained in D_R is

$2f_i / I_i$.

Theorem 14:

$$\sum_i \frac{f_i^2}{I_i} = |H|.$$

Theorem 15: If e is the identity of G and D_i is a unitary ICR

$$\sum_i \frac{\chi_i(e)\chi_i(u)^*}{I_i} = \delta(e,u)|H|.$$

Theorem 16 (Column Orthogonality): If D_i is a unitary ICR,

$$\sum_i \frac{\chi_i(u_1)\chi_i(u_2)^*}{I_i} = \delta(C_{u_1}, C_{u_2})\frac{|H|}{n_u}.$$

6. DIRECT PRODUCTS

The (inner) direct product is defined in the normal way by

$$D = D_1 \otimes D_2 \text{ if } D(g)_{ij,kl} = D_1(g)_{ik} D_2(g)_{jl}.$$

From the row orthogonality, this can be reduced directly without reference to the irreducible representations of H. We collect the interesting results in one theorem.

TABLE I. The character tables for the 58 magnetic point groups. The group or groups are given on the upper left of each table, with the ICR's beneath. In the upper middle is given the C classes with the character beneath. To the right is the intertwining number for each ICR.

(a)

$\bar{1}'$	$2'$	m'	E	I
A	A	A	1	1

(b)

	$2'/m$			E	σ_z	
		$2'/m'$		E	I	
$2/m'$			$22'2'$	E	C_{2z}	I
A	A'	A_g	A	1	1	1
B	A''	A_u	B	1	-1	1

(c)

$2m'm'$		E	C_{2z}	
	$2'm'm$	E	σ_y	I
A	A'	1	1	1
B	A''	1	-1	1

(d)

$4'$	$\bar{4}'$	E	C_{2z}	I
A	A	1	1	1
E	E	2	-2	4

(e)

$m'm'm'$			E	C_{2x}	C_{2y}	C_{2z}	
	mmm'		E	C_{2z}	σ_y	σ_x	
		$m'm'm$	E	C_{2z}	I	σ_z	I
A	A_1	A_g	1	1	1	1	1
B_3	A_2	A_u	1	1	-1	-1	1
B_2	B_1	B_g	1	-1	1	-1	1
B_1	B_2	B_u	1	-1	-1	1	1

(f)

$4'22'$	E	C_{2xz},C_{2y}	C_{2z}	I
A	1	1	1	1
E	2	0	-2	2
B_1	1	-1	1	1

(g)

$4'/m$	E	C_{2z}	I	σ_2	I
A_g	1	1	1	1	1
A_u	1	1	-1	-1	1
E_g	2	-2	2	-2	4
E_u	2	-2	-2	2	4

(h)

	$\bar{4}'2m'$		E	C_{2x},C_{2y}	C_{2z}	
$4'mm'$		$\bar{4}'2'm$	E	σ_x σ_y	C_{2z}	I
A_1	A	A_1	1	1	1	1
E	E	E	2	0	-2	2
A_2	B_1	A_2	1	-1	1	1

TABLE I (*Continued*).

(i)

$4^1/mmm$	E	C_{2x},C_{2y}	C_{2z}	I	σ_x,σ_y	σ_z	I
A_g	1	1	1	1	1	1	1
E_g	2	0	-2	2	0	-2	2
B_{1g}	1	-1	1	1	-1	1	1
A_u	1	1	1	-1	-1	-1	1
E_u	2	0	-2	-2	0	2	2
B_{1u}	1	-1	1	-1	1	-1	1

(j)

42^12^1	E	C_2^z	C_{4z}^+	C_{4z}^-	I
A	1	1	1	1	1
B	1	-1	1	-1	1
^1E	1	-1	i	$-i$	1
^2E	1	-1	$-i$	i	1

(k)

$4/m^1$		E	C_{4z}^+,C_{4z}^-	C_{2z}	
	$4^1/m^1$	E	S_{4z}^+,S_{4z}^-	C_{2z}	I
A	A	1	1	1	1
B	B	1	-1	1	1
E	E	2	0	-2	2

(l)

$4m^1m^1$		E	C_{4z}^+	C_{2z}	C_{4z}^-	
	42^1m^1	E	S_{4z}^-	C_{2z}	S_{4z}^+	I
A	A	1	1	1	1	1
B	B	1	-1	1	-1	1
^1E	^1E	1	i	-1	$-i$	1
^2E	^2E	1	$-i$	-1	i	1

(m)

$4/mm^1m^1$	E	C_{4z}^+	C_{2z}	C_{4z}^-	I	S_{4z}^-	σ_z	S_{4z}^+	I
A_g	1	1	1	1	1	1	1	1	1
B_g	1	-1	1	-1	1	-1	1	-1	1
^1E_g	1	i	-1	$-i$	1	i	-1	$-i$	1
^2E_g	1	$-i$	-1	i	1	$-i$	-1	i	1
A_u	1	1	1	1	-1	-1	-1	-1	1
B_u	1	-1	1	-1	-1	1	-1	1	1
^1E_u	1	i	-1	$-i$	-1	$-i$	1	i	1
^2E_u	1	$-i$	-1	i	-1	i	1	$-i$	1

(n)

$4/m^1m^1m^1$			E	C_{2z}	$C_{4z}^{\pm}$	$C_{2x,y}$	$C_{2a,b}$	
	$4/m^1mm$		E	C_{2z}	$C_{4z}^{\pm}$	$\sigma_{x,y}$	$\sigma_{da,b}$	
		$4^1/m^1m^1m$	E	C_{2z}	$S_{4z}^{\pm}$	$C_{2x,y}$	$\sigma_{da,b}$	I
A_1	A_1	A_1	1	1	1	1	1	1
A_2	A_2	A_2	1	1	1	-1	-1	1
B_1	B_1	B_1	1	1	-1	1	-1	1
B_2	B_2	B_2	1	1	-1	-1	1	1
E	E	E	2	-2	0	0	0	1

TABLE I (*continued*).

(o)

$32'$	$3m'$	E	C_3^+	C_3^-	I
A	A	1	1	1	1
${}^1E$	${}^1E$	1	ω	ω^*	1
${}^2E$	${}^2E$	1	ω^*	ω	1

(p)

$\bar{6}'$	$6'$	$\bar{3}'$	E	$C_3^{\pm}$	I
A	A	A	1	1	1
E	E	E	2	-1	2

(q)

$\bar{3}m'$	E	C_3^+	C_3^-	I	S_6^-	S_6^+	I
A_g	1	1	1	1	1	1	1
${}^1E_g$	1	ω	ω^*	1	ω	ω^*	1
${}^2E_g$	1	ω^*	ω	1	ω^*	ω	1
A_u	1	1	1	-1	-1	-1	1
${}^1E_u$	1	ω	ω^*	-1	$-\omega$	$-\omega^*$	1
${}^2E_u$	1	ω^*	ω	-1	$-\omega^*$	$-\omega$	1

(r)

$6'/m'$	E	$C_3^{\pm}$	I	$S_6^{\pm}$	I
A_g	1	1	1	1	1
E_g	2	-1	2	-1	2
A_u	1	1	-1	-1	1
E_u	2	-1	-2	1	2

(s)

$\bar{6}'m2'$		$\bar{3}'m'$			$6'mm'$	E	$C_3^{\pm}$	$\sigma_{d\,1,2,3}$	
	$\bar{6}'m'2$		$\bar{3}'m'$	$6'22'$		E	$C_3^{\pm}$	$C'_{21,2,3}$	I
A_1	A_1	A_1	A_1	A_1	A_1	1	1	1	1
A_2	A_2	A_2	A_2	A_2	A_2	1	1	-1	1
E	E	E	E	E	E	2	-1	0	1

(t)

$6'/m'm'm$	E	$C_3^{\pm}$	$C'_{21,2,3}$	I	$S_6^{\pm}$	$\sigma_{d\,1,2,3}$	I
A_{1g}	1	1	1	1	1	1	1
A_{2g}	1	1	-1	1	1	-1	1
E_g	2	-1	0	2	-1	0	1
A_{1u}	1	1	1	-1	-1	-1	1
A_{2u}	1	1	-1	-1	-1	1	1
E_u	2	-1	0	-2	1	0	1

(u)

$\bar{6}m'2'$			E	S_3^-	C_3^+	σ_h	C_3^-	S_3^+	
	$62'2'$	$6m'm'$	E	C_6^+	C_3^+	C_2	C_3^-	C_6^-	I
A'	A	A	1	1	1	1	1	1	1
${}^1E''$	${}^1E_1$	${}^1E_1$	1	$-\omega^*$	ω	-1	ω^*	$-\omega$	1
${}^2E''$	${}^2E_1$	${}^2E_1$	1	$-\omega$	ω^*	-1	ω	$-\omega^*$	1
A''	B	B	1	-1	1	-1	1	-1	1
${}^2E'$	${}^2E_2$	${}^2E_2$	1	ω	ω^*	1	ω	ω^*	1
${}^1E'$	${}^1E_2$	${}^1E_2$	1	ω^*	ω	1	ω^*	ω	1

TABLE I (*Continued*).

(v)

$6/m^1$	$6^1/m$	E	$S_3^{\pm}$	σ_h	$C_3^{\pm}$	
		E	$C_6^{\pm}$	C_2	$C_3^{\pm}$	I
A	A'	1	1	1	1	1
E_1	E''	2	1	-2	-1	2
B	A''	1	-1	-1	1	1
E_2	E'	2	1	2	-1	2

(w)

$6/mm^1m$	E	C_6^+	C_3^+	C_2	C_3^-	C_6^-	I	S_3^-	S_6^-	σ_h	S_6^+	S_3^-	I
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1
${}^1E_{1g}$	1	$-\omega^*$	ω	-1	ω^*	$-\omega$	1	$-\omega^*$	ω	-1	ω^*	$-\omega$	1
${}^2E_{1g}$	1	$-\omega$	ω^*	-1	ω	$-\omega^*$	1	$-\omega$	ω^*	-1	ω	$-\omega^*$	1
B_g	1	-1	1	-1	1	1	1	-1	1	-1	1	-1	1
${}^2E_{2g}$	1	ω	ω^*	1	ω	ω^*	1	ω	ω^*	1	ω	ω^*	1
${}^1E_{2g}$	1	ω^*	ω	1	ω^*	ω	1	ω^*	ω	1	ω^*	ω	1
A_u	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1
${}^1E_{1u}$	1	$-\omega^*$	ω	-1	ω^*	$-\omega$	-1	ω^*	$-\omega$	1	$-\omega^*$	ω	1
${}^2E_{1u}$	1	$-\omega$	ω^*	-1	ω	$-\omega^*$	-1	ω	$-\omega^*$	1	$-\omega$	ω^*	1
B_u	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	1
${}^2E_{2u}$	1	ω	ω^*	1	ω	ω^*	-1	$-\omega$	$-\omega^*$	-1	$-\omega$	$-\omega^*$	1
${}^1E_{2u}$	1	ω^*	ω	1	ω^*	ω	-1	$-\omega^*$	$-\omega$	-1	$-\omega^*$	$-\omega$	1

(x)

$6^1/mm^1m$	$6/m^1m^1m$	$6/m^1mm$	E	σ_h	$C_3^{\pm}$	$S_3^{\pm}$	C'_{2i}	σ_{vi}	
			E	C_{2z}	$C_3^{\pm}$	$C_6^{\pm}$	C'_{2i}	C''_{2i}	
			E	C_{2z}	$C_3^{\pm}$	$C_6^{\pm}$	σ_{di}	σ_{vi}	I
A'_1	A_1	A_1	1	1	1	1	1	1	1
A'_2	A_2	A_2	1	1	1	1	-1	-1	1
A''_1	B_1	B_2	1	-1	1	-1	1	-1	1
A''_2	B_2	B_1	1	-1	1	-1	-1	1	1
E''	E_1	E_1	2	-2	-1	1	0	0	1
E'	E_2	E_2	2	2	-1	-1	0	0	1

(y)

m^13	E	C_{2m}	$C_{3j}^{\pm}$	I
A	1	1	1	1
E	2	1	-1	2
T	3	-1	0	1

(z)

$\bar{4}^13m^1$	4^132^1	E	C_{2m}	C_{3j}^-	C_{3j}^+	I
A	A	1	1	1	1	1
${}^1E$	${}^1E$	1	1	ω	ω^*	1
${}^2E$	${}^2E$	1	1	ω^*	ω	1
T	T	3	-1	0	0	1

(aa)

$m3m^1$	E	C_{2m}	C_{3j}^-	C_{3j}^+	I	σ_m	S_{6j}^+	S_6	I
A_g	1	1	1	1	1	1	1	1	1
${}^1E_g$	1	1	ω	ω^*	1	1	ω	ω^*	1
${}^2E_g$	1	1	ω^*	ω	1	1	ω^*	ω	1
T_g	3	-1	0	0	3	-1	0	0	1
A_u	1	1	1	1	-1	-1	-1	-1	1
${}^1E_u$	1	1	ω	ω^*	-1	-1	$-\omega$	$-\omega^*$	1
${}^2E_u$	1	1	ω^*	ω	-1	-1	$-\omega^*$	$-\omega$	1
T_u	3	-1	0	0	-3	1	0	0	1

TABLE I (*Continued*).

(bb)

$m'3m'$	$m'3m$	E	$C_{3j}^{\pm}$	C_{2m}	C_{2p}	$C_{4m}^{\pm}$	
		E	$C_{3j}^{\pm}$	C_{2m}	σ_{dp}	$S_{4m}^{\pm}$	I
A_1	A_1	1	1	1	1	1	1
A_2	A_2	1	1	1	−1	−1	1
E	E	2	−1	2	0	0	1
T_1	T_1	3	0	−1	−1	1	1
T_2	T_2	3	0	−1	1	−1	1

Theorem 17:

(a) If D_i, D_j, and D_k are ICR's and

$$D_i \otimes D_j = \oplus \sum_k d_{ij}^k D_k.$$

Then

$$d_{ij}^k = \frac{1}{I_k |H|} \sum_{C_u} n_u \chi_i(u)\chi_j(u)\chi_k(u)^*.$$

(b) If **0** is the identity ICR,

$$d_{ii^*}^0 = I_i.$$

(c) If d_{ijk}^0 is the multiplicity of **0** in $D_i \otimes D_j \otimes D_k$ then

$$d_{ijk}^0 = d_{ij}^{k*} I_k.$$

This difference between the double and triple product is of great importance in developing a Racah algebra for such groups.[8]

Symmetrized and antisymmetrized squares are necessary in dealing with a number of fermions or bosons; symmetrized cubes are used in magnetic phase transitions[9]; symmetrized, antisymmetrized and mixed symmetry cubes separate out the permutation properties of the $3jm$ symbols. These can all be distinguished by character tests. For completeness we summarize them here. The notation used is $\{\lambda\}$, where $\{\lambda\}$ is a Young diagram of S_n.

(a) $\chi_{\{2\}}(u) = \frac{1}{2}([\chi(u)]^2 + \chi(u^2))$,

(b) $\chi_{\{1^2\}}(u) = \frac{1}{2}([\chi(u)]^2 - \chi(u^2))$,

(c) $\chi_{\{3\}}(u) = \frac{1}{6}([\chi(u)]^3 + 3\chi(u^2)\chi(u) + 2\chi(u^3))$,

(d) $\chi_{\{1^3\}}(u) = \frac{1}{6}([\chi(u)]^3 - 3\chi(u^2)\chi(u) + 2\chi(u^3))$,

(e) $\chi_{\{21\}}(u) = \frac{1}{3}([\chi(u)]^3 - \chi(u^3))$.

The row orthogonality now allows a direct reduction of these powers without use of tables relating these powers to the linear subgroup.

7. CONCLUSION

In this paper it has been shown that the powerful concept of a character table applies to finite magnetic groups as well as linear groups. The only added complexity is the simple intertwining number. The character table will expedite calculations as well as helping to show that corepresentation theory can stand upright without leaning on representation theory for most of its results.

To make the theory more concrete, the character tables for the fifty-eight magnetic point groups are given. They have been adapted from the tables of Cracknell[7] and all notations are the same as there.

APPENDIX: DESCENT IN SYMMETRY TO THE LINEAR SUBGROUP

The results obtained so far have been done without any reference to the representation theory of linear groups. It is known, however, that there are strong relations between the ICR's of G and the irreducible representations (IR's) of H.[1] These are generally shown by *ascent* in symmetry where the ICR's of G and their properties are determined by the IR's of H. From the methods developed earlier, we now reverse this and derive these relations by *descent* from G to H. No new results are demonstrated—rather the interplay between Schur's lemmas for linear and nonlinear groups is shown.

First we fix notation, and give the row orthogonality of an ICR of G subduced to H. Let D be a unitary ICR of G and Δ the possibly reducible representation of H obtained by descent to H. χ, the character of D on H, is also the character of Δ. Row orthogonality then gives

$$\sum_u \chi(u)\chi(u)^* = I\,|H|$$

with I the intertwining number of D. Each type of ICR is now considered in turn.

Type (a): Since I equals one, Δ is an IR. Setting

$$P = D(a_0)$$

for an arbitrary fixed element of $G - H$,

$$\Delta(a_0^2) = D(a_0)D(a_0)^* = PP^*$$

and

$$\Delta(a_0 u a_0^{-1}) = D(a_0 u a_0^{-1}) = P\Delta(u)^* P^+.$$

The ICR matrices are then given by

$$D(u) = \Delta(u) \text{ and } D(a) = D(a a_0^{-1} a_0) = \Delta(a a_0^{-1})P.$$

The IR satisfies the character test

$$\sum_a \chi_\Delta(a^2) = |H|.$$

For the other two types of ICR, Δ is reducible. To gain the results given by other authors[6] we consider the special case in which a unitary transformation has been applied to D so that Δ is in completely reduced form.

Type (b): As the intertwining number is four, Δ is reducible to either $\Delta_1 \oplus \Delta_1$ or $\Delta_1 \oplus \Delta_2 \oplus \Delta_3 \oplus \Delta_4$. This second possibility soon leads to a contradiction for by Schur's lemma for linear groups any $M \in \mathbf{m}$ must be

$$\begin{pmatrix} z_1 I & & & 0 \\ & z_2 I & & \\ & & z_3 I & \\ 0 & & & z_4 I \end{pmatrix}.$$

Schur IV shows then that $M \in \mathbf{m}'$ is purely imaginary so that M_1, M_2, *and* $M_1 M_2$ are purely imaginary. This is the required contradiction and so $\Delta = \Delta_1 \oplus \Delta_1$, i.e.,

$$D(u) = \Delta(u) = \begin{pmatrix} \Delta_1(u) & 0 \\ 0 & \Delta_1(u) \end{pmatrix}.$$

By Schur's lemma for linear groups applied to $M \in \mathbf{m}$

$$M = \begin{pmatrix} z_1 I & z_2 I \\ z_3 I & z_4 I \end{pmatrix}.$$

But by Schur II, $M + M^+ = \lambda I$ and $(M - M^+)^2 = -\mu^2 I$. Hence

$$M = x_1\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + x_2\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} + x_3\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} + x_4\begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}.$$

This in turn imposes restrictions on $D(a)$ as $MD(a) = D(a)M^*$. Choosing an arbitrary element $a_0 \in G - H$ gives

$$D(a_0) = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix},$$

$$D(a) = \begin{pmatrix} 0 & \Delta_1(aa_0^{-1})P \\ -\Delta_1(aa_0^{-1})P & 0 \end{pmatrix},$$

$$\Delta_1(a_0^2) = -PP^*,$$

$$\Delta_1(a_0 u a_0^{-1}) = P\Delta_1(u)^* P^+,$$

and

$$\sum_a \chi_{\Delta_1}(a^2) = -|H|.$$

Type (c): The intertwining number is now two so Δ is equivalent to $\Delta_1 \oplus \Delta_2$ with $\Delta_1 \not\equiv \Delta_2$. If Δ is in completely reduced form

$$\Delta(u) = \begin{pmatrix} \Delta_1(u) & 0 \\ 0 & \Delta_2(u) \end{pmatrix},$$

the same reasoning as before gives $M \in \mathbf{m}$ as

$$M = x_1\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + x_2\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.$$

For fixed arbitrary $a_0 \in G - H$,

$$D(a_0) = \begin{pmatrix} 0 & P_1 \\ P_2 & 0 \end{pmatrix},$$

$$D(a) = \begin{pmatrix} 0 & \Delta_1(aa_0^{-1})P_1 \\ \Delta_2(aa_0^{-1})P_2 & 0 \end{pmatrix},$$

$$\Delta_1(a_0^2) = P_1 P_2^* \text{ and } \Delta_2(a_0^2) = P_2 P_1^*,$$

and

$$\Delta_1(a_0 u a_0^{-1}) = P_1 \Delta_2(u)^* P_1^+,$$

$$\Delta_1(a_0 u a_0^{-1}) = P_2 \Delta_2(u)^* P_2^+,$$

from which the character test follows:

$$\sum_u \chi_{\Delta_1}(a^2) = \sum_a \chi_{\Delta_2}(a^2) = 0.$$

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