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On the symmetries of the 6*j* symbol

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The 6*j* tensor for compact groups is shown to transform as a basis vector for the identity representation of the permutation group S_4. This allows character theory to be used to determine the minimum number of independent components and a projection operator to determine the relations between components—the symmetry properties.

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1. INTRODUCTION

Following the theoretical work of Derome and Sharp,[1] 3*jm* and 6*j* tensors may be defined for any compact linear[2] or compact grey[3,4] group. The 3*jm* tensor, which has been developed from the 3*j* symbol of Wigner, reduces a triple product of (co) representations to the identity 1 of the group. In addition to the generalized azimuthal quantum numbers m_1, m_2, m_3, the 3*jm* tensor also depends on a multiplicity index r which spans a range equal to the multiplicity of 1 in the triple product of (co) representations $j_1 \otimes j_2 \otimes j_3$. The symmetry properties of the 3*jm* tensor generalize[5] from those of the Wigner 3*j* symbol: the simple phase factors are replaced by permutation matrices in the multiplicity indices. When the three representations are equivalent these matrices form a representation of the permutation group S_3 and are completely determined (to within a unitary transformation) by the properties of the group. When only two are equivalent, two of the matrices generate a representation of S_2. These again are completely determined (understood: to within a unitary transformation) whereas the others are highly arbitrary. When none are equivalent the permutation symmetry is that of the trivial group S_1 and the matrices are highly arbitrary.

The 6*j* tensor[1,4] is defined as a certain invariant product of four 3*jm* tensors. It depends on the four multiplicity indices but not on the generalized azimuthal quantum numbers. The permutation properties of the 6*j* tensor depend on the permutation matrices of the four 3*jm* tensors and, with the addition of some $1-j$ phase factors, are completely determined by these matrices. This produces some rather extreme cases: when the 3*j* symmetries are arbitrary, so are the 6*j* symmetries; on the other hand, when all the representations are equivalent, these are completely determined by group properties.

In this paper a unified method of dealing with all cases is developed. The theory is straightforward and is given in Sec. 2. It shows that a set of permuted 6*j* tensors transforms as a basis vector for the totally symmetric representation [4] of the permutation group S_4. Character theory may then be used to determine the minimum number of independant components required and a projection operator to find relations between components. These may be used as replacements for the symmetries induced from the 3*jm* tensors, which were found to be quite unmanageable in the presence of [21] symmetry.[4] Sections 2 and 3 illustrate this theory for a number of cases.

A knowledge of the papers by Derome and Sharp[1,5] or of Butler[2] is assumed and their notations are used. For simplicity only representations of linear groups are used, although all results given here apply equally to the corepresentations of grey groups.

2. THEORY

The 6*j* tensor is defined by[1,2]

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{r_1r_2r_3r_4} = (j_1j_5j_6)_{r_1m_1}{}^{m_5}{}_{m_6}(j_4j_2j_6)_{r_2m_4m_2}{}^{m_6} \times (j_4j_5j_3)_{r_3}^{m_4}{}_{m_5m_3}(j_1j_2j_3)_{r_1}^{m_1m_2m_3}.$$

The symmetry properties of the 6*j* tensor are found from the symmetry properties of the 3*jm* tensors, and are collected in Table I. Thus for example from the first entry,

$$\begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5^* & j_4^* & j_6^* \end{Bmatrix}_{s_2s_1s_3s_4} = \phi_{j_4}\phi_{j_5}\phi_{j6}\, m((12)\,j_1j_5^*j_6)_{s_1r_1}\, m((12)\,j_4j_2j_6^*)_{s_2r_2} \times m((12)\,j_4^*j_5j_3)_{s_3r_3}\, m((12)\,j_1^*j_2^*j_3^*)_{s_4r_4} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{r_1r_2r_3r_4},$$

where $m((12)\,j_1j_5^*j_6)_{s_1r_1}$ is the permutation matrix given by

$$(j_5^*j_1j_6)_{s_1m_5m_1m_6} = m((12)\,j_1j_5^*j_6)_{s_1r_1}(j_1j_5^*j_6)_{r_1m_1m_5m_6}$$

and so on, and ϕ_{j_i} is the $1-j$ phase factor for j_i. By examining the ordering of the multiplicity indices, it may be seen that each entry in the table corresponds uniquely to a permutation in S_4, and this is used to label the entries.

Given any 6*j* tensor

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{r_1r_2r_3r_4},$$

it is related to other 6*j* tensors by Table I. The number of distinct tensors depends on the irreducible representations j_1 to j_6. Thus, if all are inequivalent there are 24 distinct tensors, whereas if they are all equivalent with real character there is only 1. Each component of each of these tensors may be considered as a basis vector for a vector space V of dimension equal to the product of the four 3*j* multiplicities (M) times the number of distinct 6*j* tensors (M'). We write

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_{M'},$$

where each U_i has as basis vectors the M components of the ith 6*j* tensor. A representation of the permutation group S_4 may now be defined over V by the entries of Table I: each permutation π maps the basis of U_i onto the basis of U_j to give the (j, i) block $d^{ji}(\pi)$ of the matrix $D(\pi)$. For example,

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when all the representations are inequivalent, the division into the subspaces U_i may be taken as

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}, \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5^* & j_4^* & j_6^* \end{Bmatrix}, \begin{Bmatrix} j_3 & j_1 & j_2 \\ j_6 & j_4 & j_5 \end{Bmatrix}, \begin{Bmatrix} j_1 & j_3 & j_2 \\ j_4^* & j_6^* & j_5^* \end{Bmatrix}, \cdots$$

This gives for the block structure of $D((12))$

$$\mathrm{D}((12)) = \begin{pmatrix} 0 & d^{12}((12)) & 0 & 0 & \cdots \\ d^{21}((12)) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & d^{34}((12)) & \cdots \\ 0 & 0 & d^{43}((12)) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Each block $d^{ji}(\pi)$ is then determined directly from the entry for π in Table I by relating U_j back to U_i:

$$\begin{aligned} &d^{21}((12))_{s_2s_1s_3s_4,\, r_1r_2r_3r_4} \\ &\quad = \phi_{j_4}\phi_{j_5}\phi_{j_6} m((12)\, j_1 j_5^* j_6)_{s_1r_1}\ m((12)\, j_4 j_2 j_6^*)_{s_2r_2} \\ &\quad \times m((12)\, j_4^* j_5 j_3)_{s_3r_3}\ m((12)\, j_1^* j_2^* j_3^*)_{s_4r_4}, \\ &d^{12}((12))_{s_2s_1s_3s_4,\, r_1r_2r_3r_4} \\ &\quad = \phi^*_{j5}\phi^*_{j4}\phi^*_{j6}\ m((12)\, j_2 j_4 j_6^*)_{s_1r_1}\ m((12)\, j_5^* j_1 j_6)_{s_2r_2} \\ &\quad \times m((12)\, j_5 j_4^* j_3)_{s_3r_3}\ m((12)\, j_2^* j_1^* j_3^*)_{s_4r_4}, \end{aligned}$$

and so on. By the multiplicative properties of the $3j$ permutation matrices[1,2,5] it readily follows that the matrices $D = \{D(\pi) : \pi \epsilon S_4\}$ form a representation of S_4. Generally, of course, it is reducible.

We now write the basis vectors of V as a column vector

$$\mathbf{v} = \begin{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{r_1r_2r_3r_4} \\ \begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5^* & j_4^* & j_6^* \end{Bmatrix}_{r_1r_2r_3r_4} \\ \vdots \end{pmatrix}.$$

It follows easily that

$$D(\pi)\mathbf{v} = \mathbf{v}.$$

Hence $\mathbf{v}$ transforms as a basis vector for any totally symmetric component [4] of D. The number of such basis vectors equals the multiplicity $n_{[4]}$ of the irreducible representation [4] in D, and so the character of D completely determines the minimum number of components of a $6j$ tensor required to determine all others. The actual relations between components requires a knowledge of these basis vectors, which can be found by use of the projection operator

$$P = \Sigma_{\pi\epsilon S_4} D(\pi),$$

and these may be used to replace the symmetry properties given in Table I.

This approach is rather long-winded when the symmetry properties are simple (e.g., in a simply reducible group). However, when the $3j$ permutation matrices possess a more complex symmetry this method has been found to drastically reduce the labor involved in using the symmetry properties. In Secs. 3 and 4 examples are given to show how the method is used.

3. EXAMPLE

The 24 S_4 permutations map one $6j$ tensor into itself only when all $6j$-values are equal to j, and when $j \equiv j^*$. Setting $\phi_j = 1$ as in the quasi-ambivalent case, the permutation matrices simplify and as representative of each class we have

$$\begin{aligned} &D((12))_{s_2s_1s_3s_4,\, r_1r_2r_3r_4} \\ &\quad = m((12)\, jjj)_{s_1r_1}\ m((12)\, jjj)_{s_2r_2}\ m((12)\, jjj)_{s_3r_3}\ m((12)\, jjj)_{s_4r_4}, \\ &D((123))_{s_2s_3s_1s_4,\, r_1r_2r_3r_4} \\ &\quad = m((123)\, jjj)_{s_1r_1}\ m((123)\, jjj)_{s_2r_2}\ m((123)\, jjj)_{s_3r_3}, m((123)\, jjj)_{s_4r_4}, \\ &D((12)(34))_{s_2s_1s_4s_3,\, r_1r_2r_3r_4} \\ &\quad = m(I, jjj)_{s_1r_1}\ m(I, jjj)_{s_2r_2}\ m(I, jjj)_{s_3r_3}\ , m(I, jjj)_{s_4r_4}, \end{aligned}$$

and

$$\begin{aligned} &D((1234))_{s_2s_3s_4s_1,\, r_1r_2r_3r_4} \\ &\quad = m((13)\, jjj)_{s_1r_1}\ m((13)\, jjj)_{s_2r_2}\ m((13\, jjj)_{s_3r_3},\ m((13)\, jjj)_{s_4r_4}. \end{aligned}$$

In this case the matrices $\{m(\sigma, jjj) : \sigma \epsilon S_3\}$ form a representation of S_3 and thus the trace $\chi_D(\pi)$ of each matrix $D(\pi)$ may be found in terms of the trace $\chi_{jjj}(\sigma)$ of $m(\sigma, jjj)$. Inspection of Table I shows that $\chi_D(\pi)$ is a class function of S_4 in terms of the class function $\chi_{jjj}(\sigma)$ of S_3. This gives for the character vector χ_D

	$\{1^4\}$	$\{1^2 2\}$	$\{13\}$	$\{2^2\}$	$\{4\}$
χ_D	$(\chi_{jjj}\{1^3\})^4$	$\chi_{jjj}\{1^3\}(\chi_{jjj}\{12\})^2)$	$\chi_{jjj}\{1^3\}\chi_{jjj}\{3\}$	$(\chi_{jjj}\{1^3\})^2$	$\chi_{jjj}\{1^3\}$

The multiplicity $n_{[4]}$ follows directly from this. It may be cast into different forms by letting

$$\chi_{jjj}(\sigma) = n_{[3]}\chi_{[3]}(\sigma) + n_{[21]}\chi_{[21]}(\sigma) + n_{[1^3]}\chi_{[1^3]}(\sigma), \qquad (3.2)$$

which gives

$$\begin{aligned} n_{[4]} = (n_{[3]} + 2n_{[21]} + n_{[1^3]})\ \{&(n_{[3]} + 2n_{[21]} + n_{[1^3]})^3 \\ &+ 6(n_{[3]} - n_{[1^3]^2})^2 + 11n_{[3]} - 2n_{[21]} + 11n_{[1^3]} + 6\}/24 \end{aligned} \qquad (3.3)$$

or, by writing χ_{jjj} in terms of the character χ_j of the representation j,

$$\begin{aligned} n_{[4]} = \frac{1}{24|G|^4}\int_G [\chi_j(u)]^3\, du \Big\{ \Big(\int_G \chi_j(u)^3\, du \Big)^3 + 6|G| \\ \times \Big(\int_G \chi_j(u^2)\chi_j(u)\, du \Big)^2 + |G|^2 \int_G 3[\chi_j(u)]^3 \\ + 8\chi_j(u)^3\, du\ + 6|G|^3 \Big\}. \end{aligned} \qquad (3.4)$$

The meaning of these equations may be illustrated by taking special cases. For example, if $n_{[21]} = n_{[1^3]} = 0$,

$$n_{[4]} = (n_{[3]}/24)(n_{[3]} + 1)(n_{[3]} + 2)(n_{[3]} + 3).$$

Thus if the $3j$ multiplicity is 2, the $6j$ tensor has 16 components, of which only 5 are independent. These are of course

$$\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1111}, \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1112}, \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1122},$$
$$\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1222}, \text{ and } \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{2222}.$$

TABLE I. Permutations of the $6j$ tensor.[a]

S_4 permutation	$a\ b\ c\ d$	$j_1'\ j_2'\ j_3'\ j_4'\ j_5'\ j_6'$	Φ	σ
(12)	2 1 3 4	$j_2\ j_1\ j_3\ j_5^*\ j_4^*\ j_6^*$	$\phi_{j_4}\phi_{j_5}\phi_{j_6}$	(12)
(13)	3 2 1 4	$j_3\ j_2\ j_1\ j_6^*\ j_5^*\ j_4^*$	$\phi_{j_4}\phi_{j_5}\phi_{j_6}$	(13)
(14)	4 2 3 1	$j_1^*\ j_6^*\ j_5\ j_4\ j_3\ j_2^*$	$\phi_{j_1}\phi_{j_2}\phi_{j_6}$	(23)
(23)	1 3 2 4	$j_1\ j_3\ j_2\ j_4^*\ j_6^*\ j_5^*$	$\phi_{j_4}\phi_{j_5}\phi_{j_6}$	(23)
(24)	1 4 3 2	$j_6\ j_2^*\ j_4^*\ j_3^*\ j_5\ j_1$	$\phi_{j_2}\phi_{j_3}\phi_{j_4}$	(13)
(34)	1 2 4 3	$j_5^*\ j_4\ j_3^*\ j_2\ j_1^*\ j_6$	$\phi_{j_1}\phi_{j_3}\phi_{j_5}$	(12)
(123)	2 3 1 4	$j_2\ j_3\ j_1\ j_5\ j_6\ j_4$	1	(123)
(124)	2 4 3 1	$j_6^*\ j_1^*\ j_5\ j_3^*\ j_4^*\ j_2$	$\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}$	(132)
(134)	3 2 4 1	$j_5\ j_6^*\ j_1^*\ j_2\ j_3^*\ j_4^*$	$\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}$	(123)
(132)	3 1 2 4	$j_3\ j_1\ j_2\ j_6\ j_4\ j_5$	1	(132)
(142)	4 1 3 2	$j_2^*\ j_6\ j_4^*\ j_5^*\ j_3\ j_1^*$	$\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}$	(123)
(143)	4 2 1 3	$j_3^*\ j_4\ j_5^*\ j_6^*\ j_1\ j_2^*$	$\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}$	(132)
(234)	1 3 4 2	$j_6\ j_4^*\ j_2^*\ j_3\ j_1^*\ j_5^*$	$\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}$	(132)
(243)	1 4 2 3	$j_5^*\ j_3^*\ j_4\ j_2^*\ j_6^*\ j_1$	$\phi_{j_2}\phi_{j_3}\phi_{j_5}\phi_{j_6}$	(123)
(12)(34)	2 1 4 3	$j_4^*\ j_5^*\ j_3\ j_1^*\ j_2^*\ j_6$	$\phi_{j_1}\phi_{j_2}\phi_{j_4}\phi_{j_5}$	I
(13)(24)	3 4 1 2	$j_4^*\ j_2^*\ j_6\ j_1^*\ j_5^*\ j_3$	$\phi_{j_1}\phi_{j_5}\phi_{j_4}\phi_{j_2}$	I
(14)(23)	4 3 2 1	$j_1^*\ j_5\ j_6^*\ j_4^*\ j_2\ j_3^*$	$\phi_{j_1}\phi_{j_3}\phi_{j_4}\phi_{j_6}$	I
(1234)	2 3 4 1	$j_6^*\ j_5\ j_1^*\ j_3\ j_2^*\ j_4$	$\phi_{j_1}\phi_{j_2}\phi_{j_6}$	(13)
(1243)	2 4 1 3	$j_4\ j_3^*\ j_5^*\ j_1^*\ j_6\ j_2$	$\phi_{j_1}\phi_{j_3}\phi_{j_5}$	(23)
(1324)	3 4 2 1	$j_5\ j_1^*\ j_6^*\ j_2^*\ j_4\ j_3$	$\phi_{j_1}\phi_{j_2}\phi_{j_6}$	(12)
(1342)	3 1 4 2	$j_4^*\ j_6\ j_2^*\ j_1\ j_3^*\ j_5$	$\phi_{j_2}\phi_{j_3}\phi_{j_4}$	(23)
(1423)	4 3 1 2	$j_2^*\ j_4^*\ j_6\ j_5\ j_1\ j_3^*$	$\phi_{j_2}\phi_{j_3}\phi_{j_4}$	(12)
(1432)	4 1 2 3	$j_3^*\ j_5^*\ j_4\ j_6\ j_2\ j_1^*$	$\phi_{j_1}\phi_{j_3}\phi_{j_5}$	(13)

[a] $\begin{Bmatrix} j_1' & j_2' & j_3' \\ j_4' & j_5' & j_6' \end{Bmatrix}_{s_a s_b s_c s_d} = \Phi m(\sigma, j_1 j_5^* j_6)_{s_1 r_1} m_2(\sigma, j_4 j_2 j_6^*)_{s_2 r_2} m_3(\sigma, j_4^* j_5 j_3)_{s_3 r_3} m_4(\sigma, j_1^* j_2^* j_3^*) \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_{r_1 r_2 r_3 r_4}.$

Similarly, if $n_{[3]} = n_{[1^3]} = 0$ and $n_{[21]} = 1$, there are 16 components, of which only 1 is required. The representation $j = [321]$ of the symmetric group S_6 is of interest in that it was the first nonsimple phase representation discovered.[5] Here $n_{[3]} = 2$, $n_{[21]} = 1$, and $n_{[1^3]} = 1$. Hence there are 35 independent $6j$ components out of a total of 625.

The equations above hold whatever the choice of basis in the $3j$ multiplicity space. However, in most practical cases it will almost certainly be chosen to block diagonalize the permutation matrices according to symmetry type. For example we may write

$$m(\sigma_{jjj}) = \begin{pmatrix} m_1^{[\lambda_1]}(\sigma) & & 0 \\ & m_2^{[\lambda_2]}(\sigma) & \\ 0 & & \ddots \end{pmatrix}$$

or

$$\begin{pmatrix} m^{[3]}(\sigma)\, I_1 & & 0 \\ & m^{[1^3]}(\sigma)\, I_2 & \\ 0 & & m^{[21]}(\sigma) I_3 \end{pmatrix},$$

where $m_i^{[\lambda]}(\sigma)$ is irreducible. Since we are about to introduce a large number of different multiplicity values we shall adopt the first form as it easily allows identification of one or a pair of multiplicity values with i in $m_i^{[\lambda]}$. The argument is independent of the form, which is primarily notational.

The block diagonalization of these matrices allows an easy partial reduction of the representation D. Given four sets of multiplicity values corresponding to the ordered set i_1, i_2, i_3, i_4, the S_4 permutations transform these sets among themselves and form a closed subspace under S_4. The dimension of this subspace equals the product of the orders of each multiplicity set times the number of distinct permutations of i_1, i_2, i_3, and i_4. (The method in miniature is the same as the method applied to full $6j$ tensors.) The character depends on the matrices m_i and the equality or inequality of i_1, i_2, i_3, and i_4. Consider one case: $i_1 = i_2 = i_3 \neq i_4$. There are four permuted sets of multiplicity values corresponding to $i_1 i_1 i_1 i_4$, $i_1 i_1 i_4 i_1$, $i_1 i_4 i_1 i_1$, and $i_4 i_1 i_1 i_1$. A typical matrix over this subspace is

$$D'((12))_{js_1s_2s_3s_4,\, ir_1r_2r_3r_4}$$
$$= \begin{pmatrix} m_1^{[\lambda_1]}((12))_{s_1r_1}\, m_1^{[\lambda_1]}((12))_{s_2r_2} & 0 & 0 & 0 \\ \times m_1^{[\lambda_1]}((12))_{s_3r_3}\, m_4^{[\lambda_4]}((12))_{s_4r_4} & m_1^{[\lambda_1]}((12))_{s_1r_1}\, m_1^{[\lambda_1]}((12))_{s_2r_2} & 0 & 0 \\ 0 & \times m_4^{[\lambda_4]}((12))_{s_3r_3}\, m_1^{[\lambda_1]}((12))_{s_4r_4} & 0 & m_1^{[\lambda_1]}((12))_{s_1r_1}\, m_4^{[\lambda_4]}((12))_{s_2r_2} \\ 0 & 0 & m_4^{[\lambda_4]}((12))_{s_1r_1}\, m_1^{[\lambda_1]}((12))_{s_2r_2} & \times m_1^{[\lambda_1]}((12))_{s_3r_3}\, m_1^{[\lambda_1]}((12))_{s_4r_4} \\ 0 & 0 & \times m_1^{[\lambda_1]}((12))_{s_3r_3}\, m_1^{[\lambda_1]}((12))_{s_4r_4} & 0 \end{pmatrix}.$$

TABLE II. The character vector χ' in terms of the character vectors $\chi_i^{[\lambda_i]}$ of the reduced matrices $m_i^{[\lambda_i]}$ when all j values are equal.

	$\{1^4\}$	$\{1^2 2\}$	$\{13\}$	$\{2^2\}$	$\{4\}$
$i_1=i_2=i_3=i_4$	$(\chi_1^{[\lambda_1]}\{1^3\})^4$	$\chi_1^{[\lambda_1]}\{1^3\}(\chi_1^{[\lambda_1]}\{12\})^2$	$\chi_1^{[\lambda_1]}\{1^3\}\chi_1^{[\lambda_1]}\{3\}$	$(\chi_1^{[\lambda_1]}\{1^3\})^2$	$\chi_1^{[\lambda_1]}\{1^3\}$
$i_1=i_2=i_3\neq i_4$	$4(\chi_1^{[\lambda_1]}\{1^3\})^3\chi_4^{[\lambda_4]}\{1^3\}$	$2\chi_1^{[\lambda_1]}\{1^3\}\chi_1^{[\lambda_1]}\{12\}\chi_4^{[\lambda_4]}\{12\}$	$\chi_1^{[\lambda_1]}\{1^3\}\chi_4^{[\lambda_4]}\{3\}$	0	0
$i_1=i_2\neq i_3=i_4$	$6(\chi_1^{[\lambda_1]}\{1^3\})^2\chi_3^{[\lambda_3]}\{1^3\})^2$	$\chi_1^{[\lambda_1]}\{1^3\}(\chi_3^{[\lambda_3]}\{12\})^2$ $+\chi_3^{[\lambda_3]}\{1^3\}(\chi_1^{[\lambda_1]}\{12\})^2$	0	$2\chi_1^{[\lambda_1]}\{1^3\}\chi_3^{[\lambda_3]}\{1^3\}$	0
$i_1=i_2\neq i_3\neq i_4\neq i_1$	$12(\chi_1^{[\lambda_1]}\{1^3\})^2\chi_3^{[\lambda_3]}\{1^3\}$ $\times\chi_4^{[\lambda_4]}\{1^3\}$	$2\chi_1^{[\lambda_1]}\{1^3\}\chi_3^{[\lambda_3]}\{12\}\chi_4^{[\lambda_4]}\{12\}$	0	0	0
none equal	$24\chi_1^{[\lambda_1]}\{1^3\}\chi_2^{[\lambda_2]}\{1^3\}$ $\times\chi_3^{[\lambda_3]}\{1^3\}\chi_4^{[\lambda_4]}\{1^3\}$	0	0	0	0

from which the character is

$$\chi'((12)) = 2\chi_{jjj}^{[\lambda_1]}(\mathrm{I})\,\chi_{jjj}^{[\lambda_1]}((12))\,\chi_{jjj}^{[\lambda_4]}((12))$$

or

$$\chi'\{1^2 2\} = 2\chi_{jjj}^{[\lambda_1]}\{1^3\}\,\chi_{jjj}^{[\lambda_1]}\{12\}\,\chi_{jjj}^{[\lambda_4]}\{12\}.$$

A complete list of character vectors is given in Table II. By substitution of the various S_3 characters into this the $n_{[4]}$ multiplicities may be readily calculated. When none of i_1, i_2, i_3, or i_4 are equal, $n_{[4]}$ is the product of the multiplicity ranges corresponding to the labels i_1, i_2, i_3, and i_4. When all are equal $n_{[4]}$ is easily shown to be one for all of $[\lambda_1] = [3]$, [21], and $[1^3]$. The remaining cases are given in Tables III–V. Some of these may be compared to results in the literature. The tables of Butler[6] which extend the earlier work of Butler and Wybourne,[7] give all components of the $6j$ tensor

$$\begin{Bmatrix} T & T & T \\ T & T & T \end{Bmatrix}_{r_1 r_2 r_3 r_4},$$

where T is the three-dimensional simple phase representation of the tetrahedral group. The $3j$ permutation matrix is equivalent to $[3]\oplus[1^3]$, and Tables III and IV and equation (3.3) are in agreement with their results.

One which has not appeared before is when $i_1 = i_2 = i_3 = i_4$ and the symmetry $[\lambda] = [21]$. Using the or dering of components 1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222, the projection operator is easily found from which the single [4] basis vector is

$$(9\,0\,0\,3\,0\,3\,3\,0\,0\,3\,3\,0\,3\,0\,0\,9)^T,$$

giving

$$\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1111} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1122} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1212} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{1221} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{2112} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{2121} = 3\begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{2211} = \begin{Bmatrix} j & j & j \\ j & j & j \end{Bmatrix}_{2222},$$

with all other components zero.

4. FURTHER EXAMPLES

A closed set of three $6j$ tensors under S_4 permutations is formed when one of the $6j$ tensors is

$$\begin{Bmatrix} j_1 & j_1 & j_2 \\ j_1 & j_1 & j_2 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

and $j_1 \equiv j_1^*, j_2 \equiv j_2^*$. The $1-j$ phase factors are $\phi_{j_1} = \pm 1, \phi_{j_2} = \pm 1$. Each $3jm$ tensor is formed of the triple $(j_1 j_1 j_2)$ and hence only those S_2 permutations of the two equal j values are determined by group theory. By the same techniques as before, typical matrices over the basis

$$\begin{Bmatrix} j_1 & j_1 & j_2 \\ j_1 & j_1 & j_2 \end{Bmatrix}, \quad \begin{Bmatrix} j_1 & j_2 & j_1 \\ j_1 & j_2 & j_1 \end{Bmatrix}, \quad \begin{Bmatrix} j_2 & j_1 & j_1 \\ j_2 & j_1 & j_1 \end{Bmatrix}$$

are

TABLE III. The multiplicity $n_{[4]}$ using reduced matrices when $i_1 = i_2 = i_3 \neq i_4$ for all j values equal.

$[\lambda_1]$	$[\lambda_4]$	Total number of components	Number $n_{[4]}$ of independent components
[3]	[3]	4	1
[3]	[21]	8	0
[3]	$[1^3]$	4	0
[21]	[3]	32	2
[21]	[21]	64	2
[21]	$[1^3]$	32	2
$[1^3]$	[3]	4	0
$[1^3]$	[21]	8	0
$[1^3]$	$[1^3]$	4	1

TABLE IV. The multiplicity $n_{[4]}$ using reduced matrices when $i_1 = i_2 \neq i_3 = i_4$ for all j values equal.

$[\lambda_1]$	$[\lambda_3]$	Total number of components	Number $n_{[4]}$ of independent components
[3]	[3]	6	1
[3]	[21]	24	2
[3]	$[1^3]$	6	1
[21]	[21]	96	5
[21]	$[1^3]$	24	2
$[1^3]$	$[1^3]$	6	1

$$D((12))_{js_2s_1s_3s_4,\, ir_1r_2r_3r_4} = \begin{pmatrix} m((12)j_1j_1j_2)_{s_1r_1}\, m((12)j_1j_1j_2)_{s_2r_2} & 0 & 0 \\ \times m((12)j_1j_1j_2)_{s_3r_3}\, m((12)j_1j_1j_2)_{s_3r_3} & & \\ 0 & 0 & m((12)j_2j_1j_1)_{s_1r_1}\, m((12)j_2j_1j_1)_{s_2r_2} \\ & & \times m((12)j_2j_1j_1)_{s_3r_3}\, m((12)j_2j_1j_1)_{s_4r_4} \\ 0 & m((12)j_1j_2j_1)_{s_1r_1}\, m((12)j_1j_2j_1)_{s_2r_2} & 0 \\ & \times m((12)j_1j_2j_1)_{s_3r_3}\, m((12)j_1j_2j_1)_{s_4r_4} & \end{pmatrix}$$

and

$$D((12)(34))_{js_2s_1s_4s_3,\, ir_1r_2r_3r_4} = \begin{pmatrix} m(I,j_1j_1j_2)_{s_1r_1}\, m(I,j_1j_1j_2)_{s_2r_2} & 0 & 0 \\ \times m(I,j_1j_1j_2)_{s_3r_3}\, m(I,j_1j_1j_2)_{s_4r_4} & & \\ 0 & m(I,j_1j_2j_1)_{s_1r_1}\, m(I,j_1j_2j_1)_{s_2r_2} & 0 \\ & \times m(I,j_1j_2j_1)_{s_3r_3}\, mI,j_1j_2j_1)_{s_4r_4} & \\ 0 & 0 & m(I,j_2j_1j_1)_{s_1r_1}\, m(I,j_2j_1j_1)_{s_2r_2} \\ & & \times m(I,j_2j_1j_1)_{s_3r_3}\, m(I,j_2j_1j_1)_{s_4r_4} \end{pmatrix}.$$

These combine to give the character vector

	$\{1^4\}$	$\{1^2 2\}$	$\{13\}$	$\{2^2\}$	$\{4\}$
χ_D	$3(\chi_{j_1j_1j_2}\{1^2\})^4$	$\chi_{j_1j_1j_2}\{1^2\}(\chi_{j_1j_1j_2}\{2\})^2$	0	$3(\chi_{j_1j_1j_2}\{1^2\}$	$\chi_{j_1j_1j_2}\{1^2\}(\chi_{j_1j_1j_2}\{2\})^2$

As in Sec. 3 this may be broken down into subrepresentations according to the block diagonalization of the $3jm$ permutation matrices. No new ideas are involved.

An interesting case occurs for the $6j$ tensor

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{Bmatrix},$$

with $j_1 \equiv j_1^*, j_2 \equiv j_2^*, j_3 \equiv j_3^*$. S_4 permutations generate only six tensors, while the $3jm$ permutation matrices are essentially arbitrary. However, in this case it is easy to see that the only nontrivial elements of S_4 whose character does not vanish belong to the class $\{2^2\}$, i.e., those which interchange a pair on the top row with the corresponding pair on the bottom. The elements of this class all use the identity permutation of the $3jm$ tensors and hence the character is completely determined from a knowledge of the multiplicities alone:

	$\{1^4\}$	$\{1^2 2\}$	$\{13\}$	$\{2^2\}$	$\{3\}$
χ_D	$6(\chi_{j_1j_2j_3}\{1\}^4$	0	0	$6(\chi_{j_1j_2j_3}\{1\}^2$	0

For example in the gray tetrahedral group the multiplicity of A in the product $U' \otimes E' \otimes E$ is two.[3] This gives 5 independent components from a total of 16, no matter what form the $3j$ permutation matrices are in.

By way of contrast, the $6j$ tensor

$$\begin{Bmatrix} j_1 & j_1 & j_2 \\ j_1 & j_1^* & j_1 \end{Bmatrix}_{s_1s_2s_3s_4}$$

with $j_1 \not\equiv j_1^*$ and $j_2 \not\equiv j_2^*$, contains a wealth of information as each $3jm$ permutation matrix is determined by either S_2 or S_3 symmetries. However, none of this is used in the character (apart from multiplicities) as the S_4 permutations generate twenty-four different $6j$ tensors. The only non-vanishing character is in the class $\{1^4\}$ where it is

$$24\ \chi_{j_1j_1j_1}\{1^3\}\ \ \chi_{j_1j_1j_1^*}\{1^2\}\ \ \chi_{j_1^*j_1^*j_2}\{1^2\}\ \ \chi_{j_1^*j_1^*j_2^*}\{1^2\}.$$

The S_2 and S_3 information is of course used in the projection operator.

This is merely a representative sample of possible $6j$ tensors. All others may be dealt with in the same manner and it is not difficult to calculate projection operators for each.

TABLE V. The multiplicity $n_{[4]}$ using reduced matrices when $i_1 = i_2 \neq i_3 \neq i_4 \neq i_1$ for all j values equal.

$[\lambda_1]$	$[\lambda_3]$	$[\lambda_4]$	Total number of components	Number $n_{[4]}$ of independent components
[3]	[3]	[3]	12	1
[3]	[3]	[21]	24	1
[3]	[3]	$[1^3]$	12	0
[3]	[21]	[21]	48	2
[3]	[21]	$[1^3]$	24	1
[3]	$[1^3]$	$[1^3]$	12	1
[21]	[3]	[3]	48	3
[21]	[3]	[21]	96	4
[21]	[3]	$[1^3]$	48	1
[21]	[21]	[21]	192	8
[21]	[21]	$[1^3]$	96	4
[21]	$[1^3]$	$[1^3]$	48	3
$[1^3]$	[3]	[3]	12	1
$[1^3]$	[3]	[21]	24	1
$[1^3]$	[3]	$[1^3]$	12	0
$[1^3]$	[21]	[21]	48	2
$[1^3]$	[21]	$[1^3]$	24	1
$[1^3]$	$[1^3]$	$[1^3]$	12	1

5. CONCLUSION

With the basic theory of $3jm$ and $6j$ tensors for compact groups given by Derome and Sharp,[1,5] the major problem in applying Racah algebra methods appears to be the computational one of actually finding these tensors for specific groups. Butler and Wybourne[8] have developed a recursive technique which has been successfully applied to a number of simple phase representations.[6,7,9–13] Many groups, however, have nonsimple phase representations[14,15] and if $6j$ symbols are to be calculated efficiently for these cases it will prove important to know how many components need to be calculated. The method developed here will allow an easy determination of this number. A simple projection operator can be used to give other components.

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