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On the 3*j* symmetries

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The symmetry properties of the 3*jm* tensor for any finite or compact linear group are discussed using a wreath product construction. This is shown to provide a complete group theoretic explanation for all symmetry properties whether "essential" or "arbitrary." The link with the similar—but distinct—method of inner plethysms is considered.

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1. INTRODUCTION

Many aspects of the so-called Wigner–Racah algebra of angular momentum have been generalized to arbitrary groups, with the most complete generalizations being to compact and finite groups. While the Clebsch–Gordan series, coupling coefficients, and the Wigner–Eckart theorem now form standard components of applied group theory, not as much work has appeared on the 3*jm* tensor based on the various $3j, V, \bar{V}$ symbols of Wigner, Racah, Fano, and Racah, etc. This is largely due to the complexities in dealing both with the permutational properties of the tensor and with its relation to the coupling coefficient. The permutation properties are the reason for dealing with the 3*jm* tensor as they allow condensed tabulation and easier manipulation in equations, whereas the relation to the coupling coefficient allows the tensor to be actually used, mainly through the Wigner–Eckart theorem.

These problems were essentially solved by Derome and Sharp[1,2] in 1965 and 1966, with the first paper detailing (amongst other things) the relation between the 3*jm* and coupling coefficient tensors and the second, the symmetry properties. Their results have formed the basis for further work of both a theoretical and a computational nature.[3–16] However, judging by the number of papers appearing on coupling coefficients either without any mention of permutation properties or with some complex convention, their results are not sufficiently widely known. To improve this and also to make exact the connection with Littlewood's algebra of plethysms used by some authors,[17,18] we give in this paper an alternative derivation of the symmetries of the 3*jm* tensor. The material details a conceptual approach to the problem rather than a more efficient calculation method. Thus while a method is stated for producing symmetrized 3*jm* tensors it is unlikely that it will be used except for special classes of groups (most promising candidate: the symmetric group S_n?).

First, some background. The *coupling coefficient* is defined to be the tensor which reduces an inner direct product of two irreducible representations (irreps) $j_1 \otimes j_2$ of a group G to a third irrep of G. The 3*jm* tensor may be defined in a number of ways, but the cleanest is probably the one used by Fano and Racah[19] for SU(2) which is to reduce the *triple* product $j_1 \otimes j_2 \otimes j_3$ to the trivial irrep 1_G. The invariance of the modulus of the 3*jm* tensor under permutations follows very easily from this for G finite or compact (in addition, it forms the only really workable definition for groups containing antilinear operators[10]). The problem of relating the two tensors may be tackled in two ways: juggle the double product reduction until it becomes a triple product which introduces the 1*jm* or Wigner tensor relating j_3 to j_3^*, or expand the triple product into two double products which introduces the 2*jm* tensor reducing $j_3^* \otimes j_3$ to 1_G. Whatever, we take this problem as solved[3] and merely note three points: (a) The double product leads to the coupling coefficient and is not particularly appropriate for discussing 3*jm* permutations. (b) Relating the two tensors is not trivial as it involves the use of a third tensor. (c) The 3*jm* tensor possesses a (weak) orthogonality property through reducing the triple product, whereas the coupling coefficient possesses a stronger one by reducing only the double product. This strong orthogonality may be transferred to the 3*jm* tensor through (b).

In the Derome and Sharp approach to the 3*jm* symmetries, all possible 3*jm* tensors for the various permutations of irreps are taken and then relations sought between them. This gives a set of permutation matrices called 3*j* tensors [not to be confused with Wigner's 3*j* symbol which is a 3*jm* tensor for SU(2). In SU(2) the 3*j* tensors are just phase factors. A complete list of and explanations for this nomenclature is given in the Appendix]. By counting up the number of independent matrices and exploring their properties, Derome and Sharp were able to detail those symmetry properties which are essential and those which are arbitrary. To some extent this matrix work can be given a group interpretation by noting that the 3*j* matrices generate representations of S_3, S_2, or S_1, but for the last two cases when not all irreps are equivalent this is not sufficient to explain group theoretically all the permutuation properties.

In this paper a complete derivation for all cases is given by transferring the permutations of the 3*jm* tensor to where they act equally naturally but without regard to equivalence or inequivalence of irreps of G, namely, to the direct product group $G \times G \times G$. The permutation action of S_3 on elements of the triple product group defines a semidirect or wreath product group $\Gamma = (G \times G \times G) \circledS S_3 = G \wr S_3$. This group is discussed in the next section and its irreps are dealt with there and in the following section. These irreps are labelled

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quite naturally by irreps of G and of S_1, S_2, or S_3, with the correspondence being the same as for the $3j$ permutation matrices. However, this correspondence arises at an earlier stage.

The $3jm$ tensor is obtained by reducing irreps $j_1 \times j_2 \times j_3$ of $G \times G \times G$ to $1_{G'}$ in the diagonal subgroup $G' = \text{diag}\ G \times G \times G$; in a similar manner we obtain the $[\lambda]\ r - 3jm$ tensor by reducing irreps of Γ to $1_{\Gamma'}$ in the subgroup $\Gamma' = G' \circledS S_3$ (Sec. 4). Section 5 deals with the permutation properties of the tensor by further descent to S_3 and the relation to the symmetrized $3jm$ tensor by subduction to G' instead.

The method discussed here is not that of the plethysm algebra, although for certain cases there is a strong link. This is discussed in Sec. 6. The paper closes with a possible (though probably trivial) generalization of the symmetrized $3jm$ tensor.

Only linear groups are dealt with; as for linear/antilinear groups the direct product $j_1 \times j_2$ of two irreducible corepresentations is not generally irreducible in the direct product group.[20] The consequences of this will be discussed elsewhere.

A tensor notation is used with implied summation over repeated indices. An inner index (m) in a tensor $T^m{}_n$ corresponds to the row index for the matrix T, while an outer index(n) corresponds to the column index of the matrix. While the "niceties" of this notation are not generally used, the Hermitian adjoint of a unitary tensor $U^m{}_{m'}$ is written $U^{m'}{}_m$ when there is no risk of confusion. The notation is used because quite generally in this area some tricky points get obscured in the simple $3j$ or $\bar{V}$ notations.

2. THE GROUP $G \wr S_3$

Consider a finite or compact group G of elements u with irreducible representations (irreps) j. In order to form inner direct products $j_1 \otimes j_2 \otimes \cdots \otimes j_n$ of G, one way is to first find the irreps $j_1 \times j_2 \times \cdots \times j_n$ of the outer direct product group $G \times G \times \cdots \times G$ and then restrict this group to the diagonal subgroup diag $G \times G \times \cdots \times G$. In discussing the $3jm$ tensor it is possible to use the double product $G \times G$, but this leads directly to the coupling coefficient rather than the $3jm$, and cleaner results are obtained by considering the triple product $G \times G \times G$. The irreps of $G \times G \times G$ are just $j_1 \times j_2 \times j_3$, and on restriction to $G' = \text{diag}\ G \times G \times G$ are generally reducible. The $3jm$ tensor is nothing more than that part of the unitary matrix which reduces these representations to $1_{G'}$, the trivial representation of G'.

The permutation properties of the $3jm$ tensor were derived by Derome and Sharp by considering S_3 permutations directly on the tensor, but we observe that S_3 acts quite naturally on $G \times G \times G$ by permuting elements $(u_1 u_2 u_3)$. [Strictly, elements of $G \times G \times G$ should be written (u_1, u_2, u_3) but no confusion should arise by omitting the commas.] This action is sufficient to produce the semidirect product group $\Gamma = (G \times G \times G) \circledS S_3$. Definitions of the semidirect product vary in the literature, so we state explicitly the conventions used here. A permutation π is given in the cycle notation, with usual product [e.g., $(12)(123) = (23)$]. Each $\pi \in S_3$ acts automorphically on $G \times G \times G$ by permuting positions (not indices). Thus $(123)\,(u_a u_b u_c) = (u_b u_c u_a)$. The combination law for the permutations is then $\pi_1(\pi_2(u_a u_b u_c)) = (\pi_2 \pi_1) \times (u_a u_b u_c)$. This gives

$$\Gamma = \{(\pi, u_1 u_2 u_3):\ \pi \in S_3,\ u_i \in G\}$$

as a semidirect or wreath product group $\Gamma = (G \times G \times G) \circledS S_3 = G \wr S_3$ with multiplication

$$(\pi_1, u_1 u_2 u_3)(\pi_2, v_1 v_2 v_3) = (\pi_1 \pi_2, \pi_2(u_1 u_2 u_3) \cdot (v_1 v_2 v_3)).$$

To use this group for deriving the $3jm$ symmetries, we next need its irreps. One of the quickest ways of finding them is to start with irreps $j_1 \times j_2 \times j_3$ of $G \times G \times G$ and lift them to Γ by a two-stage process. For the first stage, consider the little group $\mathrm{L}(j_1 \times j_2 \times j_3)$ of $j_1 \times j_2 \times j_3$ in Γ. This is also a semidirect product $(G \times G \times G) \circledS S_n$, where S_n is a subgroup of S_3, and Jansen and Boon (Ref. 21, pp. 157–160) have described the process whereby $j_1 \times j_2 \times j_3$ and the irreps of S_n yield directly the irreps of the little group. These may then be induced to Γ by the usual process and these are irreducible in Γ. Further, all irreps of Γ may be obtained by this process. The details are not particularly exciting and we give the results only. Some notational aspects of the following theorem require explanation: in component form the matrix of $(u_1 u_2 u_3)$ is $j_1(u_1)^{m_1}{}_{n_1} j_2(u_2)^{m_2}{}_{n_2}\, j_3(u_3)^{m_3}{}_{n_3}$. This may be abbreviated to

$$j_1 j_2 j_3 (u_1 u_2 u_3)^{m_1 m_2 m_3}{}_{n_1 n_2 n_3}$$

or to

$$jjj_{(123)} (u_1 u_2 u_3)^{m_1 m_2 m_3}{}_{n_1 n_2 n_3}$$

without confusion. Secondly, given an irrep $[\lambda]$ of a (symmetric) group with matrices $\lambda(\pi)^r{}_{s'}$ induction to a larger group gives a representation with matrices $D_{[\lambda]}(\pi)^{r'r}{}_{s's}$, where the indices r and s label coset representatives of the group.

Theorem: Let $j_1 \times j_2 \times j_3$ be an irrep of $G \times G \times G$ with little group $\mathrm{L}(j_1 \times j_2 \times j_3)$ in $\Gamma = (G \times G \times G) \circledS S_3$, and $S_n = \mathrm{L}(j_1 \times j_2 \times j_3)/(G \times G \times G)$. Let $\{m_r \in S_3\}$ be coset representatives of S_n in S_3, $[\lambda]$ an irrep of S_n, and $D_{[\lambda]} = [\lambda] \uparrow S_3$. Then $D_{[\lambda]\, j_1 j_2 j_3}$ is an irrep of Γ, where

$$D_{[\lambda]\, j_1 j_2 j_3} (\pi, u_1 u_2 u_3)^{r' r m_1 m_2 m_3}{}_{s' s n_1 n_2 n_3} = D_{[\lambda]}(\pi)^{r'r}{}_{s's}\ jjj_{\pi_s^{-1}(123)} (u_1 u_2 u_3)^{\pi_r^{-1}\pi(m_1 m_2 m_3)}{}_{\pi_s^{-1}(n_1 n_2 n_3)}.$$

Furthermore, all irreps of Γ are of this form.

An alternative method of constructing these irreps is given by Kerber[22] in connection with the plethysm algebra. However, the emphasis in this is different and will be examined in Sec. 6.

3. EQUIVALENT IRREPS IN Γ

It must be admitted that while the theorem is given in a form eminently suitable for generalization, it does not display its salient features at a glance. We take the opportunity in this section to take a more graphic look, and also to give the equivalence transformations we shall allow.

Being heavily dependent on the little group, it is not surprising that the theorem breaks down into a number of cases when examined closer. We deal with each in turn.

A. j_1, j_2, and j_3 all equivalent

The little group here is equal to Γ itself, so the coset representatives are trivial and the indices r,s may be suppressed. $[\lambda]$ is an irrep of S_3 and can be [3], [21], or $[1^3]$. Generators of this irrep are

$$D_{[\lambda]j_1j_1j_1}((12),u_1u_2u_3)^{r'm_1m_2m_3}{}_{s'n_1n_2n_3} = \lambda\,((12))^{r'}{}_{s'}\,j_1j_1j_1(u_1u_2u_3)^{m_2m_1m_3}{}_{n_1n_2n_3} \tag{3.1}$$

and

$$D_{[\lambda]j_1j_1j_1}((123),u_1u_2u_3)^{r'm_1m_2m_3}{}_{s'n_1n_2n_3} = \lambda\,((123))^{r'}{}_{s'}\,j_1j_1j_1(u_1u_2u_3)^{m_2m_3m_1}{}_{n_1n_2n_3}. \tag{3.2}$$

Any equivalence transformation applied to this has the consequence of transforming $[\lambda]$, j_1, or both. However, for present purposes we are not interested in such transformations and conclude that no nontrivial transformations are allowed.

B. Exactly two of j_1, j_2, and j_3 equivalent

Without loss of generality we may take $j_1 = j_2 \neq j_3$. The little group is the $L(j_1\times j_1\times j_3) = (G\times G\times G)\circledS(S_2\times S_1)$ so $S_n = S_2$ with $[\lambda] = [2]$ or $[1^2]$ (as these irreps are one-dimensional, the indices r', s' are suppressed) and coset representatives e, (123),(132). Generators are then

$$D_{[\lambda]j_1j_1j_3}((12),u_1u_2u_3)^{rm_1m_2m_3}{}_{sn_1n_2n_3} = \lambda\,((12))\begin{pmatrix} j_1j_1j_3(u_1u_2u_3)^{m_2m_1m_3}{}_{n_1n_2n_3} & 0 & 0 \\ 0 & 0 & j_1j_3j_1(u_1u_2u_3)^{m_1m_3m_2}{}_{n_2n_3n_1} \\ 0 & j_3j_1j_1(u_1u_2u_3)^{m_3m_2m_1}{}_{n_3n_1n_2} & 0 \end{pmatrix}^{r}{}_{s} \tag{3.3}$$

and

$$D_{[\lambda]j_1j_1j_3}((123),u_1u_2u_3)^{rm_1m_2m_3}{}_{sn_1n_2n_3} = \lambda\,(e)\begin{pmatrix} 0 & 0 & j_1j_3j_1(u_1u_2u_3)^{m_2m_3m_1}{}_{n_2n_3n_1} \\ j_1j_1j_3(u_1u_2u_3)^{m_1m_2m_3}{}_{n_1n_2n_3} & 0 & 0 \\ 0 & j_3j_1j_1(u_1u_2u_3)^{m_3m_1m_2}{}_{n_3n_1n_2} & 0 \end{pmatrix}^{r}{}_{s}. \tag{3.4}$$

Again, transformations are restricted by the requirement that they do not alter bases in S_2 and G. In addition we impose the requirement that they do not alter $D_{[\lambda]j_1j_1j_3}(e,u_1u_2u_3)$. This restriction is imposed in order that the particularly simple structure of this irrep is not lost. [It is the diagonal matrix diag. $(j_1j_1j_3, j_3j_1j_1, j_1j_3j_1)(u_1u_2u_3)^{rm_1m_2m_3}{}_{sn_1n_2n_3}$.]

Schur's lemma in $G\times G\times G$ then shows that only diagonal transformations are allowed:

$$\begin{pmatrix} \exp(i\phi_1)I & 0 & 0 \\ 0 & \exp(i\phi_2)I & 0 \\ 0 & 0 & \exp(i\phi_3)I \end{pmatrix}. \tag{3.5}$$

C. None of j_1, j_2, or j_3 equivalent

For this last case the little group is trivial with $S_n = S_1$, and r', s' may again be suppressed. Using the coset representatives e, (12), (13), (23), (123), and (132), the generators are

$$D_{[1]j_1j_2j_3}((12),u_1u_2u_3)^{rm_1m_2m_3}{}_{sn_1n_2n_3} = \begin{pmatrix} 0 & j_2j_1j_3(u_1u_2u_3)^{m_2m_1m_3}{}_{n_2n_1n_3} & 0 & 0 & 0 & 0 \\ j_1j_2j_3(u_1u_2u_3)^{m_1m_2m_3}{}_{n_1n_2n_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j_2j_3j_1(u_1u_2u_3)^{m_2m_3m_1}{}_{n_2n_3n_1} \\ 0 & 0 & 0 & 0 & j_3j_1j_2(u_1u_2u_3)^{m_3m_1m_2}{}_{n_3n_1n_2} & 0 \\ 0 & 0 & 0 & j_1j_3j_2(u_1u_2u_3)^{m_1m_3m_2}{}_{n_1n_3n_2} & 0 & 0 \\ 0 & 0 & j_3j_2j_1(u_1u_2u_3)^{m_3m_2m_1}{}_{n_3n_2n_1} & 0 & 0 & 0 \end{pmatrix} \tag{3.6}$$

and

$$D_{[1]j_1j_2j_3}((123),\ u_1u_2u_3)^{rm_1m_2m_3}{}_{sn_1n_2n_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & j_2j_3j_1(u_1u_2u_3)^{m_2m_3m_1}{}_{n_2n_3n_1} \\ 0 & 0 & 0 & j_1j_3j_2(u_1u_2u_3)^{m_1m_3m_2}{}_{n_1n_3n_2} & 0 & 0 \\ 0 & j_2j_1j_3(u_1u_2u_3)^{m_2m_1m_3}{}_{n_2n_1n_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & j_3j_2j_1(u_1u_2u_3)^{m_3m_2m_1}{}_{n_3n_2n_1} & 0 & 0 & 0 \\ j_1j_2j_3(u_1u_2u_3)^{m_1m_2m_3}{}_{n_1n_2n_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j_3j_1j_2(u_1u_2u_3)^{m_3m_1m_2}{}_{n_3n_1n_2} & 0 \end{pmatrix} \tag{3.7}$$

Applying the same restrictions as in the last case, the only transformations allowed are those by diagonal matrices

$$\begin{pmatrix} \exp(i\phi_1)I & & & & & 0 \\ & \exp(i\phi_2)I & & & & \\ & & \exp(i\phi_3)I & & & \\ & & & \exp(i\phi_4)I & & \\ & & & & \exp(i\phi_5)I & \\ 0 & & & & & \exp(i\phi_6)I \end{pmatrix} \tag{3.8}$$

Before leaving this section we note that for each case we have been able to ignore a pair of indices as their range was trivial, although it did depend on which case was considered. Nevertheless, we seize the chance to discard them as in this type of work indices tend to proliferate. In the sequel such indices will be omitted wherever possible.

4. DESCENT TO $G'\circledS S_3$

We have constructed the irreps of $\Gamma = G^3 \circledS S_3$ and shown that they may be labelled as $[\lambda] j_1 j_2 j_3$, where $[\lambda]$ is an irrep of S_1, S_2, or S_3, and j_1, j_2, j_3 are irreps of G. We now restrict ourselves to $\Gamma' = G' \circledS S_3$ and consider the reduction of each irrep of Γ to the trivial irrep $1_{\Gamma'}$ of Γ'. First we deal with the multiplicity which we label by

$$m^{1_{\Gamma'}}_{[\lambda] j_1 j_2 j_3}.$$

To calculate this multiplicity the character alone is needed, and the specific matrix forms of the last section may be used. Each case is considered in turn.

When none of j_1, j_2, j_3 are equivalent, the trace of $D_{[1] j_1 j_2 j_3}(\pi, u_1 u_2 u_3)$ is only nonzero for $\pi = e$ when it is

$$\chi_{[1] j_1 j_2 j_3}(e, uuu) = 6\chi_1(u)\chi_2(u)\chi_3(u), \tag{4.1}$$

where χ_i is the character of j_i. The multiplicity of $1_{\Gamma'}$ is

$$m^{1_{\Gamma'}}_{[\lambda] j_1 j_2 j_3} = \frac{1}{|\Gamma'|}\int_G \sum_{\pi\in S_3} \chi_{[1] j_1 j_2 j_3}(\pi, uuu)\, du = \frac{6}{|\Gamma'|}\int_G \chi_1(u)\chi_2(u)\chi_3(u)\, du. \tag{4.2}$$

But $|\Gamma'| = 6|G|$, and so the right-hand side is the multiplicity of $1_{G'}$ in $j_1 \otimes j_2 \otimes j_3$, i.e.,

$$m^{1_{\Gamma'}}_{[1] j_1 j_2 j_3} = m^{1_{G'}}_{[1] j_1 j_2 j_3}. \tag{4.3}$$

When exactly two are equivalent (say $j_1 = j_2 \neq j_3$), the character is nonvanishing for more elements of Γ':

$$\chi_{[\lambda] j_1 j_1 j_3}(e, uuu) = 3\chi_{[\lambda]}(e)(\chi_1(u))^2\chi_3(u), \tag{4.4}$$

$$\chi_{[\lambda] j_1 j_1 j_3}((12), uuu) = \chi_{[\lambda]}((12))\,\chi_1(u^2)\,\chi_3(u), \tag{4.5}$$

with the same trace for $\pi = (13)$ and (23). This gives for the multiplicity of $1_{\Gamma'}$,

$$m^{1_{\Gamma'}}_{[\lambda] j_1 j_1 j_2} = \frac{1}{2|G|}\int_G \{\chi_{[\lambda]}(e)(\chi_1(u))^2\chi_3(u) + \chi_{[\lambda]}((12))\chi_1(u^2)\chi_3(u)\}\, du. \tag{4.6}$$

Now $[\lambda]$ can only be [2] or $[1^2]$, and the right-hand side may be recognized as the multiplicity of $1_{G'}$ in $(j_1 \odot [\lambda]) \otimes j_3$ (or of j_3^* in the symmetric or antisymmetric square of j_1). Thus

$$m^{1_{\Gamma'}}_{[\lambda] j_1 j_1 j_2} = m^{1_{G'}}_{[\lambda] j_1 j_1 j_2}. \tag{4.7}$$

The third case, when all j-values are equivalent, may be evaluated in the same way. Here $[\lambda]$ may be [3], $[1^3]$, or [21] and the multiplicity of $1_{\Gamma'}$ is the same as the multiplicity of $1_{G'}$ in the symmetric, antisymmetric, and mixed symmetry cubes of j, i.e.,

$$m^{1_{\Gamma}}_{[\lambda] jjj} = m^{1_G}_{[\lambda] jjj}. \tag{4.8}$$

We now *define* the $[\lambda]$ r-$3jm$ tensor to be the part of the unitary tensor which reduces $D_{[\lambda] j_1 j_2 j_3}$ to $1_{\Gamma'}$. In addition to the G irrep labels j and component labels m, it also contains an S_n irrep label $[\lambda]$ and component label r. It may be defined by the eigenvector equation

$$D_{[\lambda] j_1 j_2 j_3}(\pi, uuu)^{r'rm_1m_2m_3}{}_{s'sn_1n_2n_3}([\lambda] j_1 j_2 j_3)^{s'sn_1n_2n_3}{}_t = ([\lambda] j_1 j_2 j_3)^{r'rm_1m_2m_3}{}_t \tag{4.9}$$

for all $\pi \in S_3, u \in G$, where t is a "multiplicity label" with values from one to $m^{1_{\Gamma'}}_{[\lambda] j_1 j_2 j_3}$.

The unitary condition is

$$([\lambda] j_1 j_2 j_3)^{r'rm_1m_2m_3}{}_{t_1}([\lambda] j_1 j_2 j_3)^{t_2}{}_{r'rm_1m_2m_3} = \delta^{t_2}{}_{t_1} \tag{4.10}$$

which is a "weak orthogonality" property. In the next section we shall show that this can be tightened in each case.

5. THE SYMMETRIZED 3*jm* TENSOR

For the last stage in the derivation of the symmetrized $3jm$ tensor, consider the descents $\Gamma' \downarrow G'$ and $\Gamma' \downarrow S_3$. This section divides into three cases according to the previous section.

When all three j-values are equal, the coset representatives are trivial and the label r may be omitted from the $[\lambda]r$-$3jm$ tensor. Restriction to S_3 with $u = e$ makes Eq. (4.9) read

$$\lambda(\pi)^{r'}{}_{s'}([\lambda] j_1 j_1 j_1)^{s'\pi(m_1m_2m_3)}{}_t = ([\lambda] j_1 j_1 j_1)^{r'm_1m_2m_3}{}_t, \tag{5.1}$$

or, in more detail,

$$\lambda((12))^{r'}{}_{s'}([\lambda] j_1 j_1 j_1)^{s'm_2m_1m_3}{}_t = ([\lambda] j_1 j_1 j_1)^{r'm_1m_2m_3}{}_t \tag{5.2}$$

and

$$\lambda((123))^{r'}{}_{s'}([\lambda] j_1 j_1 j_1)^{s'm_2m_3m_1}{}_t = ([\lambda] j_1 j_1 j_1)^{r'm_1m_2m_3}{}_t. \tag{5.3}$$

Thus if $[\lambda] = [3]$, the value of the tensor is unchanged under permutation of m-values; if $[\lambda] = [1^3]$ it changes sign under tranpositions but not under cyclic permutations; if $[\lambda] = [21]$ there are *two* tensors for each t corresponding to r' equals 1 or 2. These two tensors transform as a basis for [21] and hence their transformation properties are much more complex. Many examples of such "nonsimple phase" cases are now known to occur in both the finite[23–25] and compact Lie[18] groups, although few numerical cases have appeared. Schindler and Mirman[26] have given some for the symmetric groups but apparently without complete recognition.

On descent to G', Eq. (4.9) becomes

$$j_1(u)^{m_1}{}_{n_1} j_1(u)^{m_2}{}_{n_2} j_1(u)^{m_3}{}_{n_3} ([\lambda]j_1j_1j_1)^{r'n_1n_2n_3}{}_t = ([\lambda]j_1j_1j_1)^{r'm_1m_2m_3}{}_t. \quad (5.4)$$

However, this is just one of the ways of defining the ordinary $3jm$ tensor: its columns are a complete set of independent solutions satisfying a certain orthogonality condition. The strongest orthogonality condition is that derived from the coupling coefficient,

$$(j_1j_1j_1)^{m_1m_2m_3}{}_{t'_1}(j_1j_1j_1)^{t'_2}{}_{m_1m_2m'_3} = |j_1|^{-1}\delta^{t'_2}{}_{t'_1}\delta^{m_3}{}_{m'_3}, \quad (5.5)$$

where $|j_1|$ is the dimension of j_1. (This equation holds for a sum over any pair of m-values, not just m_1 and m_2.) The columns of the $3jm$ tensor form a basis for a vector space called the multiplicity space, and any column of the $[\lambda]r$-$3jm$ tensor must therefore be expressible in terms of the $3jm$ tensor

$$([\lambda]j_1j_1j_1)^{r'm_1m_2m_3}{}_t = A_{[\lambda]}{}^{r't'}{}_t(j_1j_1j_1)^{m_1m_2m_3}{}_{t'}. \quad (5.6)$$

From the unitary condition (4.10) we already have one orthogonality property. The last two equations allow this to be tightened. Form the inner product (Hermitian adjoint)

$$\begin{aligned}([\lambda_1]j_1j_1j_1)^{r'_1m_1m_2m_3}{}_{t_1}&([\lambda_2]j_1j_1j_1)^{t_2}{}_{r'_2m_1m_2m'_3}\\ &= A_{[\lambda_1]}{}^{r'_1t'_1}{}_{t_1}A_{[\lambda_2]}{}^{t_2}{}_{r'_2t'_2}|j_1|^{-1}\delta^{m_3}{}_{m'_3}\delta^{t'_2}{}_{t'_1}\\ &= B^{r'_1t_2}{}_{r'_2t_1}|j_1|^{-1}\delta^{m_3}{}_{m'_3},\end{aligned} \quad (5.7)$$

where $B^{r'_1t_2}{}_{r'_2t_1} = A_{[\lambda_1]}{}^{r'_1t'_1}{}_{t_1}A_{[\lambda_2]}{}^{t_2}{}_{r'_2t'_1}$. This may be improved further by equating m_3 and m'_3, summing, and then utilizing Eq. (5.1):

$$\begin{aligned}B^{r'_1t_2}{}_{r'_2t_1} &= ([\lambda_1]j_1j_1j_1)^{r'_1m_1m_2m_3}{}_{t_1}([\lambda_2]j_1j_1j_1)^{t_2}{}_{r'_2m_1m_2m_3}\\ &= \lambda_1(\pi)^{r'_1}{}_{s'_1}\lambda_2(\pi^{-1})^{s'_2}{}_{r'_2}([\lambda_1]j_1j_1j_1)^{s'_1\pi(m_1m_2m_3)}{}_{t_1}\\ &\quad\times([\lambda_2]j_1j_1j_1)^{t_2}{}_{s'_2\pi(m_1m_2m_3)}\\ &= \lambda_1(\pi)^{r'_1}{}_{s'_1}\lambda_2(\pi^{-1})^{s'_2}{}_{r'_2}B^{s'_1t_2}{}_{s'_2t_1},\end{aligned} \quad (5.8)$$

i.e.,

$$\lambda_1(\pi)^{r'_1}{}_{s'_1}B^{s'_1t_2}{}_{s'_2t_1} = B^{r'_1t_2}{}_{r'_2t_1}\lambda_2(\pi)^{r'_2}{}_{s'_2}. \quad (5.9)$$

For each pair t_1,t_2, the matrix B intertwines λ_1 and λ_2 and hence by Schur's lemma is zero for $\lambda_1 \not\equiv \lambda_2$ and diagonal for $\lambda_1 = \lambda_2$. Hence

$$B^{r'_1t_2}{}_{r'_2t_1} = |\lambda_1|^{-1}\delta^{r'_1}{}_{r'_2}C^{t_2}{}_{t_1}\delta(\lambda_1,\lambda_2). \quad (5.10)$$

Substituting this into Eq. (5.7), equating r'_1 and r'_2, m_3 and m'_3, and summing invokes the orthogonality of the $[\lambda]r$-$3jm$ tensor so that C is also a delta tensor. Hence

$$\begin{aligned}([\lambda_1]j_1j_1j_1)^{r'_1m_1m_2m_3}{}_{t_1}&([\lambda_2]j_1j_1j_1)^{t_2}{}_{r'_2m_1m_2m'_3}\\ &= |\lambda_1|^{-1}|j_1|^{-1}\delta(\lambda_1,\lambda_2)\delta^{r'_1}{}_{r'_2}\delta^{t_2}{}_{t_1}\delta^{m_3}{}_{m'_3}.\end{aligned} \quad (5.11)$$

Counting dimensions, this shows that the columns of the $[\lambda]r$-$3jm$ tensors for $[\lambda] = [3]$, $[21]$, and $[1^3]$ form an orthogonal basis for the $3jm$ multiplicity space. Further, if the $[\lambda]r$-$3jm$ tensor is divided by $|\lambda|^{1/2}$, this basis satisfies the same orthogonality property as any $3jm$ tensor and hence *defines* a particular $3jm$ tensor—the "symmetrized" $3jm$ tensor. That is, if the columns of the $[\lambda]r$-$3jm$ tensor are divided by $|\lambda|^{1/2}$, they form columns (or pairs of columns when $[\lambda] = [21]$) of the $3jm$ tensor, and these columns transform as basis vectors for $[\lambda]$ when the m-values are permuted. Any other $3jm$ tensor derived by any method is related to this one by a unitary transformation in the multiplicity space. This completes the discussion for all irreps equal.

When exactly two are equal, say $j_1 = j_2 \not\equiv j_3$, the primed labels may be omitted as $[\lambda] = [2]$ or $[1^2]$ is one-dimensional. On descent to G', Eq. (4.9) gives

$$\begin{pmatrix} j_1j_1j_3(uuu)^{m_1m_2m_3}{}_{n_1n_2n_3} & 0 & 0\\ 0 & j_3j_1j_1(uuu)^{m_3m_1m_2}{}_{n_3n_1n_2} & 0\\ 0 & 0 & j_1j_3j_1(uuu)^{m_2m_3m_1}{}_{n_2n_3n_1}\end{pmatrix}^r{}_s \times([\lambda]j_1j_1j_3)^{sm_1m_2m_3}{}_t = ([\lambda]j_1j_1j_3)^{rn_1n_2n_3}{}_t. \quad (5.12)$$

This breaks into three equations, and each equation defines subspaces of $3jm$ multiplicity spaces. For r equals 1, 2, and 3, these are subspaces of the $3jm$ tensors $(j_1j_1j_3)$, $(j_3j_1j_1)$, and $(j_1j_3j_1)$, respectively. As for the last case, orthogonality of the $3jm$ tensors produces

$$\begin{aligned}([\lambda_1]j_1j_1j_3)^{1m_1m_2m_3}{}_{t_1}&([\lambda_2]j_1j_1j_3)^{t_2}{}_{1m_1m_2m'_3}\\ &= B^{1t_2}{}_{1t_1}|j_3|^{-1}\delta^{m_3}{}_{m'_3}\end{aligned} \quad (5.13)$$

with similar conditions for $r = 2,3$. Under the e and (12) permutations applied to this, Schur's lemma in S_2 gives $\delta(\lambda_1,\lambda_2)$ on the right, whereas applying (13) and (23) permutations shows

$$B^{1t_2}{}_{1t_1} = B^{2t_2}{}_{2t_2} = B^{3t_2}{}_{3t_1}.$$

Orthogonality of the $[\lambda]r$-$3jm$ tensor in addition gives

$$B^{rt_2}{}_{rt_1} = \delta^{t_2}{}_{t_1}$$

so that $1/\sqrt{3}\,([\lambda]j_1j_1j_3)$ for $[\lambda] = [2],[1^2]$ defines three particular $3jm$ tensors $(j_1j_1j_3)$,$(j_3j_1j_1)$, and $(j_1j_3j_1)$.

The permutation properties of these symmetrized $3jm$ tensors follow by descent $\Gamma'\downarrow S_3$. A (12) permutation is completely defined for the $(j_1j_1j_3)$ tensor:

$$\lambda(12)([\lambda]j_1j_1j_3)^{1m_2m_1m_3}{}_t = ([\lambda]j_1j_1j_3)^{1m_1m_2m_3}{}_t$$

so that this tensor changes sign if $[\lambda] = [1^2]$ but is left invariant if $[\lambda] = [2]$. Identical properties hold for $(j_3j_1j_1)$ under (23) transpositions, and for $(j_1j_3j_1)$ under (13) transpositions.

The effect of permutations can also be calculated for the off-diagonal elements, but it must be noted that the allowed equivalence transformations in Γ produce an arbitrariness in this. For the off-diagonal (12) elements,

$$\lambda\,((12))([\lambda\,]\,j_1 j_1 j_3)^{3m_1m_2m_3}{}_t = \exp i(\phi_3 - \phi_2)([\lambda\,]\,j_1 j_1 j_3)^{2m_1m_2m_3}{}_t$$

and for (123),

$$([\lambda\,]\,j_1 j_1 j_3)^{3m_1m_2m_3}{}_t = \exp i(\phi_3 - \phi_1)([\lambda\,]\,j_1 j_1 j_3)^{1m_1m_2m_3}{}_t,$$

$$([\lambda\,]\,j_1 j_1 j_3)^{1m_1m_2m_3}{}_t = \exp i(\phi_1 - \phi_2)([\lambda\,]\,j_1 j_1 j_3)^{2m_1m_2m_3}{}_t,$$

etc. The choices of ϕ_1, ϕ_2, and ϕ_3 are entirely upto the user, and are determined by choosing an irrep in Γ from its equivalence class.

The final case of all irreps inequivalent follows this last case very closely. When divided by $\sqrt{6}$, the $[\lambda\,]r$-$3jm$ defines six symmetrized $3jm$ tensors corresponding to the six orderings of $(j_1 j_2 j_3)$. Permutations in S_3 map these tensors onto each other with at most a change of phase. These phase factors are completely arbitrary (to within consistency imposed by multiplications) and are determined by the choice of irrep in Γ. For the irreps given explicitly in Sec. 3 all phase factors are 1 so that these symmetrized $3jm$ tensors are invariant under any permutation of j and m-values.

Particular $3jm$ tensors have thus been shown to arise naturally on reducing irreps of $(G\times G\times G)\circledS S_3$ to the trivial irrep of the subgroup (diag $G\times G\times G)\circledS S_3$. These $3jm$ tensors are "symmetrized" in that their transformation properties under permutations are as simple as possible. Any other $3jm$ tensors are related to these by unitary transformations in the multiplicity space. While there are arbitrary phase factors in some of these permutation properties, they are explained as arising from equivalence transformations in $G \wr S_3$.

6. THE METHOD OF PLETHYSMS

It was mentioned in the introduction that some authors have used Littlewood's algebra of plethysms to obtain results about $3j$ symmetries, mainly as to the existence or nonexistence of nonsimple phase irreps. As we have *not* used plethysms but something quite closely related, it is worth detailing this link. There are in fact two plethysm algebras, the "inner" and the "outer." However, as the outer plethysm algebra is really just the inner plethysm algebra for the general linear group transferred to the symmetric group via the duality between the two groups, we need only talk about inner plethysms to cover all cases.

The inner plethysm construction, as detailed for example by Kerber,[22] is quite heavily dependent on the irreps and carrier spaces of a group. Given an irrep j of G with carrier space V, the action of S_n on $V\times V\times\cdots\times V$ (n times) is defined by permutations of the basis vectors. This gives a representation of S_n over this direct product space which may be combined with the irrep j to give an irrep of $G\sim S_n$ of dimension $|j|^n$. For the case n equals 3, this is in fact the irrep of Γ we have called $D_{[3]jjj}$. The other irreps with $[\lambda\,]=[21]$ and $[1^3]$ may be obtained quite readily, but this is a "second-stage" calculation and they do not appear quite so naturally. For our other irreps when not all j-values are equivalent, the plethysm method works best in a subgroup of S_n [for example $G\wr(S_2\times S_1)$ for exactly two j-values equal] and this is not sufficient to give all $3j$ symmetries for these cases. The irreps in this subgroup may, however, be induced to $G\sim S_n$ as we have done.

The subgroup Γ' is isomorphic to $G\wr S_3$ and the plethysm method reduces representations of Γ' to $j\times[\lambda\,]$ of $G\times S_3$. In the context of $3j$ symmetries the reduction is to $1_G\times[\lambda\,]$. In our notation, this is reduction to $D_{[\lambda\,]1_G1_G1_G}$ with $[\lambda\,]=[3]$, [21], and $[1^3]$. A complete list of the multiplicities of these irreps in all irreps of Γ' is given in Table I. From there it may be seen that the multiplicity of $D_{[\lambda\,]1_G1_G1_G}$ in $D_{[3]jjj}$ is the same as the multiplicity of $1_{\Gamma'}$ in $D_{[\lambda\,]jjj}$ but that many other entries on this table (in particular when not all j-values are equivalent) are rather strange. We dispose of these cases first.

When only two j-values are equal, the natural group for inner plethysms is $G\wr(S_2\times S_1)$ and its irreps are reduced in

TABLE I. Multiplicity of irreps $[\mu]\times 1_G$ in representations $D_{[\lambda\,]j_1j_2j_3}$ of (diag $G\times G\times G)\circledS S_3$.

	$[3]\times 1_G$	$[1^3]\times 1_G$	$[21]\times 1_G$
$[3]j_1j_1j_1$	$\frac{1}{6G}\int_G \{\chi_1(u)\}^3 + 3\chi_1(u^2)\chi_1(u) + 2\chi_1(u^3)\,du$	$\frac{1}{6G}\int_G \{\chi_1(u)\}^3 - 3\chi_1(u^2)\chi_1(u) + 2\chi_1(u^3)\,du$	$\frac{1}{3G}\int_G \{\chi_1(u)\}^3 - \chi_1(u^3)\,du$
$[1^3]j_1j_1j_1$	$\frac{1}{6G}\int_G \{\chi_1(u)\}^3 - 3\chi_1(u^2)\chi_1(u) + 2\chi_1(u^3)\,du$	$\frac{1}{6G}\int_G \{\chi_1(u)\}^3 + 3\chi_1(u^2)\chi_1(u) + 2\chi_1(u^3)\,du$	$\frac{1}{3G}\int_G \{\chi_1(u)\}^3 - \chi_1(u^3)\,du$
$[21]j_1j_1j_1$	$\frac{1}{3G}\int_G \{\chi_1(u)\}^3 - \chi_1(u^3)\,du$	$\frac{1}{3G}\int_G \{\chi_1(u)\}^3 - \chi_1(u^3)\,du$	$\frac{1}{3G}\int_G 2\{\chi_1(u)\}^3 + \chi_1(u^3)\,du$
$[2]j_1j_1j_2$	$\frac{1}{2G}\int_G \{\chi_1(u)\}^2\chi_3(u) + \chi_1(u^2)\chi_3(u)\,du$	$\frac{1}{2G}\int_G \{\chi_1(u)\}^2\chi_3(u) - \chi_1(u^2)\chi_3(u)\,du$	$\frac{1}{G}\int_G \{\chi_1(u)\}^2\chi_3(u)\,du$
$[1^2]j_1j_1j_2$	$\frac{1}{2G}\int_G \{\chi_1(u)\}^2\chi_3(u) - \chi_1(u^2)\chi_3(u)\,du$	$\frac{1}{2G}\int_G \{\chi_1(u)\}^2\chi_3(u) + \chi_1(u^2)\chi_3(u)\,du$	$\frac{1}{G}\int_G \{\chi_1(u)\}^2\chi_3(u)\,du$
$[1]j_1j_2j_3$	$\frac{1}{G}\int_G \chi_1(u)\chi_2(u)\chi_3(u)\,du$	$\frac{1}{G}\int_G \chi_1(u)\chi_2(u)\chi_3(u)\,du$	$\frac{2}{G}\int_G \chi_1(u)\chi_2(u)\chi_3(u)\,du$

$G'\circledS(S_2\times S_1)\cong G\times S_2$. The reduction is then to $1_G\times[2]$ and $1_G\times[1^2]$ which only deals with the (12) permutations of $(j_1 j_1 j_3)$, the (13) permutations of $(j_1 j_3 j_1)$, or the (23) permutations of $(j_3 j_1 j_1)$. A (123) permutation cannot be used to relate the three tensors for it is simply not in the group. On the other hand, using Γ' is not correct either for it is not clear as to whether reductions should be to $1_G\times[3]$, $1_G\times[21]$, or $1_G\times[1^3]$. Inner plethysms are just not appropriate for discussing the $3j$ symmetries of this case. Similar considerations hold for none of the irreps equivalent.

When all three j-values are equal though, the multiplicities certainly tally and inner plethysms can be used for characterlike calculations. A closer investigation reveals that the plethysm reduction is something like a coupling coefficient to the $[\lambda]r$-$3jm$ tensor. We define this plethysm reduction by a $3jm$-$[\mu]r$ tensor:

$$jjj(uuu)^{\pi(m_1m_2m_3)}{}_{n_1n_2n_3} = \sum_{[\mu]\in \mathrm{Irr}(S_3)} (jjj)_{[\mu]}{}^{m_1m_2m_3}{}_{rt'}\,\mu(\pi)^r{}_s\,(jjj)_{[\mu]}{}^{st'}{}_{n_1n_2n_3} \tag{6.1}$$
$$\oplus \text{ other irreps of } G\times S_3.$$

If this equation is multiplied by $\lambda(\pi)^{r'}{}_{s'}$ and integrated over Γ' divided by its volume, the left-hand side becomes

$$([\lambda]\,jjj)^{r'm_1m_2m_3}{}_t\,([\lambda]\,jjj)^t{}_{s'n_1n_2n_3} \tag{6.2}$$

as this is the only component transforming as $1_{\Gamma'}$. On the right the only nonvanishing component is that which transforms as $1_G\times[3]$ of $G\times S_3$ which can only occur for the [3] component of $[\lambda]\otimes[\mu]$. Thus $[\mu]$ must equal $[\lambda]$ and the direct product must be reduced by a $2jm$ tensor in S_3:

$$([\lambda]\,jjj)^{r'm_1m_2m_3}{}_t\,([\lambda]\,jjj)^t{}_{s'n_1n_2n_3} = (jjj)_{[\lambda]}{}^{m_1m_2m_3}{}_{rt'}\,([\lambda][\lambda])^{rr'}([\lambda][\lambda])_{ss'} \times (jjj)_{[\lambda]}{}^{st'}{}_{n_1n_2n_3}. \tag{6.3}$$

By the orthogonality properties of all the tensors this may be recast into

$$([\lambda]\,jjj)^{r'm_1m_2m_3}{}_t = U^{t'}{}_t\,(jjj)_{[\lambda]}{}^{m_1m_2m_3}{}_{rt'}\,([\lambda][\lambda])^{rr'}, \tag{6.4}$$

where U is a unitary tensor relating bases in the multiplicity spaces. By transforming one of the $[\lambda]\,r$-$3jm$ or $3jm$-$[\mu]\,r$ tensors this may be taken as diagonal so that one tensor may be found directly from the other.

The $2jm$ tensor in Eq. (6.4) is, in general, *not* trivial. However, if the irreps of S_3 are chosen to be *orthogonal* it reduces to a tensor which merely changes columns into rows. (This is because the $1jm$ tensor is diagonal.) For this special case, which is nevertheless the most common one, we may write

$$([\lambda]\,jjj)^{r'm_1m_2m_3}{}_t = |\lambda|^{-1/2}\delta^{rr'}\,(jjj)_{[\lambda]}{}^{m_1m_2m_3}{}_{rt}$$

to give one tensor from the other.

7. A POSSIBLE GENERALIZATION

It has been shown that the permutation properties of the $3jm$ tensor may be found by reducing irreps of $\Gamma = G\wr S_3$ to the trivial irrep $1_{\Gamma'}$ in the subgroup Γ', or when all j-values are equal by reducing certain irreps of Γ to certain others in Γ'. In Table I this means we have used the first row and the first column only. A quite natural question is to ask what tensors correspond to the other entries. A formal answer is to define a tensor which reduces $D_{[\lambda]j_1j_2j_3}$ to $D_{[\mu]1_G1_G1_G}$, which for consistent terminology must go under the title of a $[\lambda]r$-$3jm$-$[\mu]r$ tensor. Such a tensor must always exist for any group even if trivially by setting $[\lambda]=[\mu], j_1=j_2=j_3=1_G$. It is unlikely that this tensor will prove of much importance as it can be expressed in terms of $3jm$ tensors for G and S_3, but if any sufficiently interesting results are discovered they will be reported.

APPENDIX

In this appendix we state the terminology which appears to be most appropriate in discussing the Wigner–Racah algebra. For more details the reader is referred to Derome and Sharp[1,2] or better the article by Butler.[3]

The *1jm tensor or Wigner tensor*. For any calculations with irreps of a group, it is assumed that the matrices of each irrep are fixed. To each irrep j there is a conjugate irrep j^* (which may of course equal j). If the complex conjugate of the matrices of j is taken then this matrix irrep will be *equivalent* to the matrices of j^*. The $1jm$ tensor is the matrix of equivalence. The "m" in the notation signifies that it is basis dependent.

The *1j phase*. By Schur's lemma, the product of the $1jm$ tensors for $j\to j^*$ and $j^*\to j$ is a scalar matrix λI. λ is the $1j$ phase. It is independent of basis so no "m" labels are included.

The *2jm tensor* is the tensor which reduces the inner product $j\otimes j^*$ to 1_G, the trivial irrep.

The *2j phase* is the phase factor arising on permuting the j- and m-values in the $2jm$ tensor.

The *coupling coefficient* is the tensor which reduces the inner product of two irreps $j_1\otimes j_2$.

The *3jm tensor* is the tensor which reduces the inner product of three irreps $j_1\otimes j_2\otimes j_3$ to 1_G.

The *3j tensor* is the permutation tensor in the multiplicity labels relating one $3jm$ tensor to another (permuted) one. It is independent of basis labels m but does depend on the three irrep labels.

The *Clebsch–Gordan series* gives the multiplicity of each irrep in the inner product $j_1\otimes j_2$. This is based on the original series for SO(3), and is *not* the coupling coefficient.

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